KE-style tableaux for a family of intuitionistic modal logics

A. Solares-Rojas, M. Sztajn, and R. O. Rodriguez

June 19, 2026

The proof system KE is a cut-based variant of tableaux, which is more efficient than these and the closely related cut-free sequent calculus. Namely, the "canonical" restriction of KE—that is *analytic* and as "mechanical" as tableau-like methods—can polynomially simulate the other two systems, while the reverse relation does not hold. Indeed, canonical KE has exponentially shorter proofs than tableaux and the cut-free sequent calculus w.r.t. the class of "truly fat" formulas in clausal form [4, Proposition 5.8]. What is more, the same class of formulas shows that tableaux cannot even polynomially simulate the truth-tables method [2]. KE's hallmark is reducing the amount of branching to a minimum by making all branches mutually exclusive. Specifically, it has a unique branching zero-premise rule that implements the Principle of Bivalence over, initially, any formula A:

$$\frac{}{\mathsf{T}\,A \mid \mathsf{F}\,A}$$

However, this rule can be restricted so as to be analytic and be applied "mechanically" without affecting completeness nor efficiency. So restricted, the rule corresponds to a non-eliminable analytic-cut rule. The rest of the rules of KE all have a non-branching format and correspond to traditional reasoning patterns such as modus ponens, modus tollens, disjunctive syllogism and its dual.

Recently, we [8] extended the KE system to intuitionistic propositional logic (**IPL**) by means of labeled signed formulas so as to mimic countermodel-search in the relational semantics. Besides inheriting KE's hallmark of reducing branching to a minimum—and unlike the currently most efficient SAT-based methods [e.g. 5]—our intuitionistic KE enjoys native interpretability within its underlying semantics and is modular in that it can naturally implement the "frame-condition to rule" method. Specifically, we import the pre-order relation into the object language via expressions of the form $c_i \preceq c_j$, called *constraints* and where the constant labels c_i and c_j intuitively denote states or worlds [see 3, 6]. In turn, this expressions are taken as side conditions in the formulation of the rules of the intuitionistic KE system, which we reproduce here:

$$\frac{\mathsf{F}\,A \vee B : c_i}{\begin{array}{l}\mathsf{F}\,A : c_i\\ \mathsf{F}\,B : c_i\end{array}} \qquad \frac{\mathsf{T}\,A \wedge B : c_i}{\begin{array}{l}\mathsf{T}\,A : c_i\\ \mathsf{T}\,B : c_i\end{array}} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \vee B : c_i\\ \mathsf{F}\,A : c_j\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{T}\,B : c_i} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \vee B : c_i\\ \mathsf{F}\,B : c_j\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{T}\,A : c_i}$$

$$\frac{\begin{array}{l}\mathsf{F}\,A \wedge B : c_j\\ \mathsf{T}\,A : c_i\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{F}\,B : c_j} \qquad \frac{\begin{array}{l}\mathsf{F}\,A \wedge B : c_j\\ \mathsf{T}\,B : c_i\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{F}\,A : c_j} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \to B : c_i\\ \mathsf{T}\,A : c_j\\ \boxed{c_i \preceq c_k \text{ and } c_j \preceq c_k}\end{array}}{\mathsf{T}\,B : c_k} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \to B : c_i\\ \mathsf{F}\,B : c_j\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{F}\,A : c_j}$$

$$\frac{\mathsf{F}\,A \to B : c_i}{\begin{array}{l}\mathsf{T}\,A : c_j\\ \mathsf{F}\,B : c_j\\ c_i \preceq c_j\end{array}}\ c_j \text{ new} \qquad \frac{\begin{array}{l}\mathsf{T}\,\neg A : c_i\\ \boxed{c_i \preceq c_j}\end{array}}{\mathsf{F}\,A : c_j} \qquad \frac{\mathsf{F}\,\neg A : c_i}{\begin{array}{l}\mathsf{T}\,A : c_j\\ c_i \preceq c_j\end{array}}\ c_j \text{ new} \qquad \frac{\begin{array}{l}\mathsf{T}\,A : c_i\\ \mathsf{F}\,A : c_j\\ \boxed{c_i \preceq c_j}\end{array}}{\times}$$

$$\frac{}{\mathsf{T}\,A : c_i \mid \mathsf{F}\,A : c_i}$$

Now, Balbiani and Gencer [1] recently put forward a uniform framework for characterizing a robust family of intuitionistic modal logics that, in the spirit of Fischer-Servi's **IK**, contains "the formulas whose standard translation in a first-order language are intuitionistically valid". The authors provided both axiomatic and semantic characterizations of these logics. In particular, a bi-relational semantics is given, where the conditions for $\Diamond$ depart from the traditional ones. Namely, $\Diamond A$ is forced at a state w in a model $(W, \preceq, R, V)$ if and only if there exists a state w' such that $w \succeq \circ R w'$ and A is forced at w', where $\succeq \circ R$ denotes the composition of the intuitionistic pre-order $\preceq$ and the modal accessibility relation R. Under this new satisfiability condition for $\Diamond$, the *Heredity Property* can be proven without requiring any confluence condition between $\preceq$ an R. For this reason, in this setting, the axiomatization of validity in the class of all models constitutes a *minimal* logic, $\mathbf{L}_{\min}$, which is in that sense analogous to its classical counterpart **K**. While incomparable neither with Wijesekera's **WK** nor **CK**, $\mathbf{L}_{\min}$ is strictly contained in **IK** that, in turn, corresponds to validity in the class of all forward and backward confluent models. Other classes of models associated with different confluence conditions between $\preceq$ and R—the latter possibly under a further condition of reflexivity, symmetry or transitivity—induce logics deductively stronger than $\mathbf{L}_{\min}$, some of which can be axiomatized and belong to a family having as basis that minimal logic.

In the line of Marin et al. [7], in this work we apply the "frame-condition to rule" method to a familiy of intuitionistic modal logics as characterized by Balbiani and Gencer's setting. Namely, we enrich our intuitionistic KE system with rules for the modalities, importing both the pre-order and the modal accessibility relation to the object language so as to obtain constraints of two forms $c_i \preceq c_j$ and $c_k\, R\, c_l$, and thus formulating a KE system that corresponds to $\mathbf{L}_{\mathtt{min}}$. That is, adding the following rules to those above:

$$\frac{\begin{array}{l}\mathsf{T}\ \Box A : c_i\\ \boxed{c_i \preceq c_j, c_j\, R\, c_k}\end{array}}{\mathsf{T}\ A : c_k} \qquad \frac{\mathsf{F}\ \Box A : c_i}{\begin{array}{l}\mathsf{F}\ A : c_k\\ c_i \preceq c_j, c_j\, R\, c_k\end{array}}\ c_j,\, c_k \text{ new}$$

$$\frac{\mathsf{T}\ \Diamond A : c_i}{\begin{array}{l}\mathsf{T}\ A : c_k\\ c_i \succeq c_j, c_j\, R\, c_k\end{array}}\ c_j,\, c_k \text{ new} \qquad \frac{\begin{array}{l}\mathsf{F}\ \Diamond A : c_i\\ \boxed{c_i \succeq c_j, c_j\, R\, c_k}\end{array}}{\mathsf{F}\ A : c_k}$$

Based on that "minimal" KE-style system, we add further rules corresponding to additional frame conditions or axioms, characterizing logics in a family that extends $\mathbf{L}_{\mathtt{min}}$. For example, for **IK** the conditions corresponding to forward and backward confluent frames are respectively translated into the additional rules:

$$\frac{\boxed{c_i \succeq c_j, c_j\, R\, c_k}}{c_i\, R\, c_l, c_l \succeq c_k}\ c_l \text{ new} \qquad \frac{\boxed{c_i\, R\, c_j, c_j \preceq c_k}}{c_i \preceq c_l, c_l\, R\, c_k}\ c_l \text{ new}$$

KE-style systems for other logics in the family are obtained by following the same method, encoding other conditions over the composition of the relations R and $\preceq$—possibly together with particular conditions on R—into inference rules manipulating constraints only. For example, adding rules corresponding respectively to up- and down-symmetric frames:

$$\frac{\boxed{c_i\ R\ c_j}}{c_j \preceq c_m, c_m\ R\ c_n, c_n \preceq c_i}\ c_m, c_n \text{ new} \qquad \frac{\boxed{c_i\ R\ c_j}}{c_j \succeq c_m, c_m\ R\ c_n, c_n \succeq c_i}\ c_m, c_n \text{ new}$$

Soundness of the resulting systems is straightforward given that the labels mechanism basically mimics the underlying bi-relational semantics. As for completeness, for the time being, we have proven it by simulating the axiomatic characterizations of each of these logics. That is, we have shown that each rule of the corresponding axiomatic system can be simulated in the respective KE system, while each axiom can be derived as a theorem therein. However, we shall discuss how the arguments for termination via countermodel extraction, given in [8] for the plain intuitionistic case, can be extended so as to cover our intuitionistic modal KE-style systems.

Devising KE-tableaux for intuitionistic modal logics following the "frame-condition to rule" method is by no means restricted to families based on $\mathbf{L}_{\mathrm{min}}$. We find Balbiani and Gencer's "minimal setting" as a particularly uniform approach to implement the method but, in principle, it can be similarly applied for devising KE-tableaux for families based on either **CK** or **WK**, which also admit of bi-relational semantics with some additional conditions. Indeed, this investigation constitutes a first example showing how our intuitionistic KE system can be modularly extended to include modalities in a natural way.

References

[1] P. Balbiani and Ç. Gencer. Intuitionistic modal logics: A minimal setting. *Studia Logica*, pages 1–68, 2026.

[2] M. D'Agostino. Are tableaux an improvement on truth-tables? *Journal of Logic, Language and Information*, 1(3):235–252, 1992.

[3] M. D'Agostino and D. Gabbay. A generalization of analytic deduction via labelled deductive systems. Part I: Basic substructural logics. *Journal of Automated Reasoning*, 13(2):243–281, 1994.

[4] M. D'Agostino and M. Mondadori. The taming of the cut. Classical refutations with analytic cut. *Journal of Logic and Computation*, 4(3):285–319, 1994.

[5] C. Fiorentini. Efficient SAT-based Proof Search in Intuitionistic Propositional Logic. In A. Platzer and G. Sutcliffe, editors, *Automated Deduction – CADE 28*, pages 217–233, Cham, 2021. Springer International Publishing.

[6] D. Galmiche and D. Méry. Semantic Labelled Tableaux for Propositional BI without $\perp$. *Journal of Logic and Computation*, 13(5):707–753, 2003.

[7] S. Marin, M. Morales, and L. Straßburger. A fully labelled proof system for intuitionistic modal logics. *Journal of Logic and Computation*, 31(3):998–1022, 2021.

[8] A. Solares-Rojas, P. Baldi, and R. O. Rodriguez. Labeled KE for intuitionistic propositional logic. Manuscript accepted in *Journal of Logic and Computation*, 2026.