Polytopological Semantics for Intuitionistic Modal Logics

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June 16, 2026

Both intuitionistic logic and the modal logics $\mathsf{K4}$ and $\mathsf{S4}$ (along with many related logics) enjoy topological semantics dating back to McKinsey and Tarski [6], predating even Kripke semantics. It is only natural to expect for intuitionistic modal logics to also enjoy a topological interpretation, and indeed there have been some efforts in this direction [3, 5, 4, 8]. However, none of these proposals simultaneously treat intuitionistic implication and the modalities via independent topologies according to the McKinsey–Tarski semantics. This would mirror the relational setting, where semantics are given by structures $\langle W, \preccurlyeq, \sqsubset \rangle$, consisting of a set equipped with two preorders, possibly satisfying additional confluence properties. A natural topological adaptation would be to equip a set with distinct topologies $\tau_{\to}$ and $\tau_{\Diamond}$. This raises the question of whether this leads to a reasonable interpretation of constructive or intuitionistic logic and what, if any, compatibility conditions on the topologies are required to obtain a well-behaved semantics. In this work, we identify suitable polytopological semantics for $\mathsf{CK4}$ and some of its extensions.

The logic $\mathsf{CK4}$ is a constructive modal logic which extends CK by the axioms $4_{\Box} := \Box\varphi \to \Box\Box\varphi$ and $4_{\Diamond} := \Diamond\Diamond\varphi \to \Diamond\varphi$. Other logics we consider are extensions of $\mathsf{CK4}$, with combinations of the following:

$$T_{\Box} := \Box\varphi \to \varphi, \qquad \mathrm{FS} := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi),$$

$$T_{\Diamond} := \varphi \to \Diamond\varphi, \qquad \mathrm{DP} := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi.$$

We then define the following well-known extensions of $\mathsf{CK4}$:

$$\mathsf{IK4} := \mathsf{CK4} + \{\mathrm{FS}, \mathrm{DP}\}, \quad \mathsf{CS4} := \mathsf{CK4} + \{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\}, \quad \mathsf{IS4} := \mathsf{CS4} + \{\mathrm{FS}, \mathrm{DP}\}.$$

Our polytopological semantics in principle involve three topologies: one for the intuitionistic implication and one for each of the two modalities. A *tritopological space* is a tuple $\mathcal{X} = \langle W^{\mathcal{X}}, \tau^{\mathcal{X}}_{\to}, \tau^{\mathcal{X}}_{\Diamond}, \tau^{\mathcal{X}}_{\Box} \rangle$, where $W^{\mathcal{X}}$ is a non-empty set and $\tau^{\mathcal{X}}_{\to}$, $\tau^{\mathcal{X}}_{\Diamond}$, and $\tau^{\mathcal{X}}_{\Box}$ are topologies over $W^{\mathcal{X}}$. A *bitopological space* is defined similarly, but omitting $\tau^{\mathcal{X}}_{\Box}$.

For a connective $\circ \in \{\to, \Diamond, \Box\}$, we define $i^{\mathcal{X}}_\circ$ and $c^{\mathcal{X}}_\circ$ to be the respective interior and closure operators with respect to $\tau^{\mathcal{X}}_\circ$. Similarly, we let $d_\circ$ and $e_\circ$ denote the Cantor derivative and co-derivative. Recall that if A is a subspace of a topological space X, then its Cantor derivative, dA, is the set of limit points of A, i.e. those $x \in X$ such that $x \in c(A \setminus \{x\})$ (where c denotes the standard topological closure). Meanwhile, the co-derivative is obtained by dualising, as $eA = X \setminus d(X \setminus A)$.

When possible, we work with bitopological rather than tritopological spaces, as these are a bit simpler. Note that if X is any set and τ, σ are topologies on X, then $\tau \cap \sigma$ is also a topology on X. Every bitopological space $\mathcal{X} = \langle W^{\mathcal{X}}, \tau^{\mathcal{X}}_\to, \tau^{\mathcal{X}}_\Diamond \rangle$ thus induces a tri-topological space by adding $\tau^{\mathcal{X}}_\Box := \tau^{\mathcal{X}}_\to \cap \tau^{\mathcal{X}}_\Diamond$.

We may then use bitopological spaces to provide semantics for extensions of $\mathsf{CS4}$ by equipping a bitopological space $\mathcal{X}$ with a valuation $\|\cdot\|$, which to each formula assigns an open subset of $W^{\mathcal{X}}$ according to the topological semantics of intuitionistic modal logic, in such a way that $\|\Box\varphi\| = i^{\mathcal{X}}_\Box \|\varphi\|$ and $\|\Diamond\varphi\| = i^{\mathcal{X}}_\to c^{\mathcal{X}}_\Diamond \|\varphi\|$. As usual, $\|\cdot\|$ is uniquely determined by its values on atomic formulas.

Given a logic Λ, say that a Λ-*space* is a polytopological space for which Λ is sound. Remarkably, any bitopological space is a $\mathsf{CS4}$-space, but stronger logics require additional assumptions on the topologies.

Definition 1. *Let $\mathcal{X} = \langle X, \tau_\to, \tau_\Diamond \rangle$ be a bitopological space. We say that $\mathcal{X}$ is:*

$\Diamond$-Coarse *iff $i_\to i_\Diamond = i_\Box i_\Diamond$, i.e., the $\tau_\to$-interior and the $\tau_\Box$-interior of any $\tau_\Diamond$-open set coincide.*

$\Diamond$-Regular *iff $c_\Diamond i_\to = i_\to c_\Diamond i_\to$, i.e. the $\tau_\Diamond$-closure of any $\tau_\to$-open set is $\tau_\to$-open.*

Then, $\mathcal{X}$ is an $\mathsf{IS4}$-space *if it is $\Diamond$-coarse and $\Diamond$-regular.*

Below, recall that a topological space is *Alexandroff* if open sets are closed under arbitrary (rather than only finite) intersections. Alternately, an Alexandroff space may be characterised as one whose open sets are the up-sets of some preorder $\preccurlyeq$, making them be closely related to Kripke frames. We may thus adapt completeness results from the literature [1, 7] to obtain the following.

Theorem 2. *If $\Lambda \in \{\mathsf{CS4}, \mathsf{IS4}\}$, then Λ is strongly complete for the class of Alexandroff bitopological Λ-spaces.*

For logics without the T axioms, we no longer obtain completeness for the class of bitopological spaces and must work with tritopological spaces. We moreover assume that $\tau_\Diamond$ and $\tau_\Box$ are T_d, i.e., they validate the 4 axioms. Alternately, T_d spaces can be characterised as those where every singleton is the intersection of a closed set and an open set. It is known that every T_d space is T_0 and every T_1 space is T_d.

In the tritopological setting, $\tau_\Box$ need not be the intersection of the other two topologies, but we do require that $e_\Box i_\to = i_\to e_\Box i_\to$ and $e_\Box i_\to = i_\Diamond e_\Box i_\to$. The notions of $\Diamond$-coarseness and $\Diamond$-regularity may be adapted by replacing closure by Cantor derivative, and thus $\mathsf{IK4}$-spaces are those in which $i_{\to}e_{\Diamond} = e_{\Box}e_{\Diamond}$ and $d_{\Diamond}i_{\to} = i_{\to}d_{\Diamond}i_{\to}$.

Semantics are defined in a similar way as before, but with $\|\Box\varphi\| = e^{\mathcal{X}}_{\Box}\|\varphi\|$ and $\|\Diamond\varphi\| = i^{\mathcal{X}}_{\to}d^{\mathcal{X}}_{\Diamond}\|\varphi\|$.

Theorem 3. *If $\Lambda \in \{\mathsf{CK4}, \mathsf{IK4}\}$, then Λ is strongly complete for the class of T_d Alexandroff tritopological Λ-spaces.*

Alexandroff spaces are obtained because these completeness results are inherited from Kripke completeness. We may similarly appeal to Kripke completeness of other extensions of $\mathsf{CK4}$ to obtain analogous results; for example, semantics for Gödel-Dummett logics are obtained by letting the intuitionistic topology be hereditarily extremally disconnected [2], yielding analogous completeness results for $\mathsf{GK4}$ and $\mathsf{GS4}$, defined by adding $(p \to q) \vee (q \to p)$ to $\mathsf{IK4}$ and $\mathsf{IS4}$, respectively.

Alexandroff spaces usually violate most (if not all) of the separation axioms typically assumed of topological spaces. One may wonder if completeness still holds for each logic when one or more of the topologies is Hausdorff or even metrizable, in the spirit of the McKinsey–Tarski theorem [6], leading to many questions that we leave open in this exploratory work. For example, while any bitopological space is a $\mathsf{CS4}$-space, even the construction of Hausdorff $\mathsf{IS4}$ spaces seems to be non-trivial. Likewise, we have considered only T_d spaces, but the (classical) d-logic of all topological spaces is $\mathsf{wK4}$, which is weaker than $\mathsf{K4}$. We do not know if its constructive analogue is topologically complete.

References

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