Gödel coding on fibrations and geminal categories*

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1 Introduction

The provability logic GL was originally motivated by formal provability, but the same GL-like pattern also appears in type-theoretic and computational settings. Notably, Kavvos [3] observes that the Gödel–Löb axiom $\Box(\Box A \to A) \to \Box A$ can be read constructively as a principle of *intensional recursion.* This suggests that there is a common categorical structure underlying both the provability modality and the intensional modality of code. Indeed, Kavvos [2] also introduces the dual-context modal calculus DGL and gives its categorical semantics via *Gödel–Löb categories,* which provide a categorical refinement of GL-like modal structures with computational content.

A different route to such structures is suggested by Ramesh's *geminal categories* [7]. Geminal categories may be seen as a categorical abstraction of the provability modality: prototypical examples are obtained from syntactic categories of formal theories. Their basic idea is to use internal categories to axiomatize Gödelian self-formalization—a theory's internal representation of its own syntax. The remarkable feature of this approach is that the Löb principle is not assumed as a part of the structure; rather, a categorical counterpart of Löb's theorem is derived from the "self-internalizing" structure by a highly non-trivial argument.

In this work, we further analyze the structure of geminal categories. Specifically, we establish a contextual generalization of Ramesh's categorical Löb theorem, a categorical counterpart of the Gödel–Löb axiom, and a detailed comparison between geminal categories and Kavvos' Gödel–Löb categories. These results are based on a self-contained reorganization of the theory of geminal categories, guided by the new notion of *code structures* on fibrations; this reorganization also substantially simplifies Ramesh's original proof of the categorical Löb theorem.

2 Preliminaries

Let $\mathcal{B}$ be a category with finite limits. Recall that a $\mathcal{B}$-internal category $\mathbb{C}$ consists of an object of objects $\mathbb{C}_{\mathrm{ob}} \in \mathcal{B}$, an object of morphisms $\mathbb{C}_{\mathrm{mor}} \in \mathcal{B}$, and structure maps satisfying the usual category axioms internally. Arrows $1 \to \mathbb{C}_{\mathrm{ob}}$ and $1 \to \mathbb{C}_{\mathrm{mor}}$ are called *global objects* and *global morphisms* of $\mathbb{C}$, respectively. We write $\Gamma\mathbb{C}$ for the ordinary category of global objects and global morphisms of $\mathbb{C}$.

Definition 1. A *geminal category* is a category $\mathcal{B}$ with chosen finite limits, equipped with a $\mathcal{B}$-internal category $\mathbb{C}$ with chosen internal finite limits, a functor $F\colon \mathcal{B} \to \Gamma\mathbb{C}$ strictly preserving finite limits, and an internal functor satisfying two equations. For brevity, we omit the conditions on this internal functor; see Ikeda [1, Definitions 4.15 and 6.5] for details.

Henceforth, we fix a geminal category and use $\mathcal{B}, \mathbb{C}$ and F for its components as above. We note that the above definition coincides with Ramesh's compact presentation of geminal categories [7, Definition 5.15], up to notation.

*This work is based on the author's Master's thesis [1] and ongoing research.

Example 2. Prototypical examples of geminal categories are given by syntactic categories of sufficiently expressive formal theories. Suppose that a theory can internally represent enough of its own syntax and provability to support the standard proofs of Gödel's incompleteness theorems and Löb's theorem. Then one obtains a geminal category by taking $\mathcal{B}$ to be the external syntactic category of the theory and $\mathbb{C}$ to be its internally represented syntactic category. The functor $F\colon \mathcal{B} \to \Gamma\mathbb{C}$ sends an external syntactic object to its internal code, and hence plays the role of Gödel coding.

At the propositional level, any geminal category provides an interpretation of the $\{\top, \wedge, \Box\}$-fragment of GL (equivalently, iGL), as well as the corresponding *contextual* extension indicated by the notation below. The propositional connectives are interpreted in $\mathcal{B}$ in the usual way:

$$[\![\top]\!] = 1, \quad [\![P \wedge Q]\!] = [\![P]\!] \times [\![Q]\!].$$

Other connectives, such as $\to, \bot, \vee$, may be added modularly when the geminal category has the corresponding structure.

Instead of introducing a formal modal calculus and its full interpretation explicitly, we use the following semantic notation to clarify the modal reading. For objects $A, B \in \mathcal{B}$, we use the contextual modal notation[1]

$$[B \vdash A] := \mathbb{C}(FB, FA) \in \mathcal{B},$$

where $\mathbb{C}(FB, FA)$ denotes the object in $\mathcal{B}$ of internal arrows from the global object FB to the global object FA in $\mathbb{C}$. Moreover, if $a\colon B \to A$ is a morphism in $\mathcal{B}$, then $Fa\colon FB \to FA$ is a global morphism of $\mathbb{C}$, equivalently, an arrow $1 \to \mathbb{C}(FB, FA)$ in $\mathcal{B}$. We write $\ulcorner a \urcorner$ for this arrow:

$$\frac{a\colon B \to A}{\ulcorner a \urcorner\colon 1 \to [B \vdash A]}$$

The usual box modality is the special case with empty context:

$$\Box A := [1 \vdash A] \cong \mathbb{C}(1_{\mathbb{C}}, FA).$$

This determines the endofunctor $\Box\colon \mathcal{B} \to \mathcal{B}$. Note that $\ulcorner a \urcorner\colon 1 \to \Box A$ for $a\colon 1 \to A$.

In Example 2, when A and B are subterminal objects (i.e., propositions), the objects $[B \vdash A]$ and $\Box A$ express, respectively, the provability of the sequent $B \vdash A$ and the provability of the proposition A. In Cartesian closed settings, one has $[B \vdash A] \cong \Box(B \Rightarrow A)$.

The non-trivial point is that the Löb principle is valid in this interpretation. Ramesh proves that every geminal category satisfies a categorified version of Löb's theorem, which can be viewed as a fixed point theorem.

Theorem 3 (Ramesh [7, Observation 5.24])**.** *For any object $A \in \mathcal{B}$ and any morphism $f : \Box A \to A$, there is a unique morphism $a\colon 1 \to A$ such that $f \circ \ulcorner a \urcorner = a$.*

3 Results

We now state the results obtained through our reorganization of the theory of geminal categories. The first is the following *contextual* generalization of Theorem 3:

Theorem 4. *For any objects $A, B \in \mathcal{B}$ and any morphism $f\colon [B \vdash A] \times B \to A$, there exists a unique morphism $a\colon B \to A$ such that $f \circ \langle \ulcorner a \urcorner \circ !, \mathrm{id}\rangle = a$.*

$$\begin{tikzcd}
B \arrow[r, "\langle \ulcorner a \urcorner \circ !, \mathrm{id}\rangle"] \arrow[dr, "a"'] & {[B \vdash A] \times B} \arrow[d, "f"] \\
& A
\end{tikzcd}$$

[1]This notation is inspired by *contextual modal type theory* [6]; the notation $[B \vdash A]$ follows Murase et al. [5].

The case $B = 1$ recovers Theorem 3; even for this special case, our proof is significantly simpler than the original one. One advantage of the contextual form is that it yields the categorical counterpart of the Gödel–Löb axiom:

Corollary 5. *For any object $A \in \mathcal{B}$, there exists a morphism $Y_A \colon [\Box A \vdash A] \to \Box A$ satisfying a certain commutative diagram (see Ikeda [1, Theorem 6.11]).*

Under the semantic reading above, this morphism is the sequent-style form of the Gödel–Löb axiom, namely $[\Box A \vdash A] \vdash \Box A$; in Cartesian closed setting, it corresponds to the usual $\Box(\Box A \to A) \to \Box A$.

Using this corollary, we further analyze the modal structure induced by the endofunctor $\Box : \mathcal{B} \to \mathcal{B}$ (see Ikeda [1, Theorem 6.16]). The resulting structure is strikingly close to Kavvos' Gödel–Löb categories [2], although a subtle mismatch remains concerning the order of certain modal and fixed-point operations. Clarifying whether this mismatch is essential requires further investigation, but the comparison suggests that geminal categories or related structures may provide categorical models for GL-related modal calculi.

4 Methods

To reorganize the theory of geminal categories, we introduce the notion of *code structures,* an additional structure on fibrations. These structures serve as a categorical abstraction of Gödel coding within fibrational semantics. Their definition is designed so that the proof of the classical *diagonal lemma* can be reproduced categorically; the resulting theorem is what we call the *fixed point theorem for fibrations with codes.* This theorem may be viewed as an *intensional* variant of Lawvere's fixed point theorem [4].

We show that any geminal category canonically induces code structures on several basic fibrations over the base category, such as codomain fibrations and representable fibrations. Notably, our main theorem is proved by two successive applications of the fixed point theorem to these fibrations. This highlights the core idea of Ramesh's original proof, which we call the *bootstrapping argument.*

This development differs from Ramesh's original one by working directly with geminal categories. Ramesh [7] introduces geminal categories in a broader study of *introspective theories,* namely theories whose models carry self-internalizing structures, and derives Theorem 3 as a consequence of that framework. Our approach bypasses the machinery of introspective theories and provides a self-contained framework for geminal categories.

5 Conclusion and future work

We have further analyzed the structure of geminal categories. Our method based on code structures on fibrations also provides a streamlined approach to geminal categories, making the framework more accessible.

A central task for future work is to resolve the remaining mismatch with Kavvos' Gödel–Löb categories and to establish a precise correspondence between geminal categories and modal type theories. The significance of this direction lies in its potential to bridge *metamathematics* and *metaprogramming:* by uncovering the categorical structures common to both, such a bridge may allow insights from provability logic to be transferred to computer science, and conversely. This would provide a step toward a unified perspective on meta- and object-level interactions in logic and computation.

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