Cover Semantics for Fitch-Style Modal Natural Deduction

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Intuitionistic modal logics typically extend intuitionistic logic with one or both of the (dual) modalities $\Box$ and $\Diamond$. In the last decades, these logics have found myriad applications in notably AI [6], database theory [14], concurrency theory [19] and the design of type systems for programming languages [16, 12, 4]. But their study is notoriously hard, as it combines the subtleties of intuitionistic logic IPL with the complexity of modalities. In this work, we are interested in the $\Box$-only logic $\mathsf{CK}_\Box$, which extends IPL with the following axiom K and *necessitation* rule:

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad\qquad \frac{\emptyset \vdash A}{\Gamma \vdash \Box A}$$

Intuitionistic propositional logic famously corresponds to the simply-typed lambda calculus, with the formulas and proofs of the logic respectively matched to the types and terms of the lambda calculus: this is the celebrated *Curry-Howard correspondence* [7, 18]. Through the extension of this correspondence, several intuitionistic modal logics have been found to correspond to *modal lambda calculi* [1, 2]. A notable Curry-Howard correspondence exists between the logic $\mathsf{CK}_\Box$ and a modal lambda calculus due to Borghuis [3]. This calculus, which can be viewed as a natural deduction system $\mathsf{FCK}_\Box$, was later given a categorical foundation and a refined presentation by Clouston [4].

The Curry-Howard correspondence between the logic $\mathsf{CK}_\Box$ and Borghuis' calculus allows for the sharing of tools: machinery to study the logic can be leveraged to investigate the lambda calculus, and vice-versa. One such tool is Kripke-style birelational semantics for $\mathsf{CK}_\Box$. By extending the usual concept of a preordered frame $(X, \sqsubseteq)$ in Kripke's semantics for IPL with a binary relation R, Kripke-style birelational semantics allows us to interpret the $\Box$ modality of $\mathsf{CK}_\Box$, for a model $\mathcal{M}$ induced by the frame, in the following way:

$$\mathcal{M}, w \Vdash \Box A \qquad \text{iff} \qquad \forall w', v.\, w \sqsubseteq w' \text{ and } w' \mathrel{R} v \text{ implies } \mathcal{M}, v \Vdash A$$

It can be shown that $\mathsf{CK}_\Box$ tightly corresponds to the local consequence relation $\models$ over these frames [15, 5], as witnessed by the following soundness-and-completeness result: $\Gamma \vdash A \;\Leftrightarrow\; \Gamma \models A$.

While one can use this semantics to study $\mathsf{FCK}_\Box$, we identify two deficiencies in this approach. First, the establishment of completeness proofs with respect to Kripke-style semantics are well-known to readily require non-constructive principles [13, 17]. While non-constructivity can be seen as problematic in itself, it also prevents the algorithmic use of the semantics critical for *normalisation by evaluation* (NbE)—a goal we wish to reach for $\mathsf{FCK}_\Box$. In our specific case, the decidability of $\mathsf{CK}_\Box$ [11] and our focus on finitary contexts should allow for a constructive proof of completeness similar to Hagemeier and Kirst's [10]. Still, this trick becomes out of reach once the logic turns undecidable, e.g. in the presence of quantifiers, making this semantics a poor foundation for the study of extensions of $\mathsf{CK}_\Box$. Second, the above Kripke-style semantics does not provide an insightful account of the rules in $\mathsf{FCK}_\Box$: it does not explain the context modifications present in the rules for $\vee$ and $\bot$, as we elaborate below.

Our work aims to develop a Beth-Kripke-Joyal-style *cover* semantics [8, 9] for $\mathsf{FCK}_\Box$ addressing both deficiencies, by allowing for constructive completeness (and NbE) while giving a semantic account of the unusual rules in $\mathsf{FCK}_\Box$.

The Proof System $\mathsf{FCK}_\Box$ The formal language of the logic $\mathsf{CK}_\Box$ consists of formulas defined inductively by propositional atoms (p, q, r, etc.), constants $\top$ and $\bot$, binary connectives $\wedge, \vee, \Rightarrow$ and a unary connective $\Box$.

$$\text{Prop} \quad A, B := p, q, r, \ldots \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \Rightarrow B \mid \Box A \qquad \text{Ctx} \quad \Gamma, \Delta := \cdot \mid \Gamma, A \mid \Gamma, \text{🔒}$$

The proof system $\mathsf{FCK}_\Box$ [4] is a sequent-style natural deduction calculus for $\mathsf{CK}_\Box$, and is defined inductively as usual using inference rules for judgements $\Gamma \vdash A$ that relate a *context* Γ to a formula A. A context Γ in a judgement of $\mathsf{FCK}_\Box$ is a finite sequence $A_1, ..., A_i, \text{🔒}, A_j, ..., A_k, \text{🔒}, A_m, ..., A_n$ consisting of sequences of formulas delimited by a constant symbol 🔒, with $\cdot$ denoting the empty sequence. The inference rules for $\mathsf{FCK}_\Box$ are given below. Observe, in particular, how the rules ($\bot$-E) and ($\vee$-E) depart from the usual formulation in IPL and allow the context in the conclusion to be extended with an arbitrary context Γ'.

$$\frac{A \in \Gamma \quad 🔒 \notin \Gamma'}{\Gamma, \Gamma' \vdash A}\ (\text{Hyp}) \qquad \frac{}{\Gamma \vdash \top}\ (\top\text{-I}) \qquad \frac{\Gamma \vdash \bot}{\Gamma, \Gamma' \vdash A}\ (\bot\text{-E})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\text{-I}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge\text{-E}_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge\text{-E}_2)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\text{-I}_2) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A, \Gamma' \vdash C \quad \Gamma, B, \Gamma' \vdash C}{\Gamma, \Gamma' \vdash C}\ (\vee\text{-E})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow\text{-I}) \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow\text{-E}) \qquad \frac{\Gamma, 🔒 \vdash A}{\Gamma \vdash \Box A}\ (\Box\text{-I}) \qquad \frac{\Gamma \vdash \Box A \quad 🔒 \notin \Gamma'}{\Gamma, 🔒, \Gamma' \vdash A}\ (\Box\text{-E})$$

Cover Semantics for $\mathsf{FCK}_\Box$ A cover system $C = (W, \sqsubseteq, \lhd, R)$ is a tuple consisting of a set W of worlds, a preorder $\sqsubseteq$ on W, a *covering* relation $\lhd \subseteq W \times \mathcal{P}(W)$, and a *modal accessibility* relation $R \subseteq W \times W$, all subject to certain conditions. We say w is *refined by* w' or w' *refines* w when $w \sqsubseteq w'$, and say w is *covered by* α or α *covers* w when $w \lhd \alpha$. Following the usual practice, we say w *can access* v or v is *accessible from* w when $w\ R\ v$. We may intuitively understand a world w as a "state of knowledge", the refinement $w \sqsubseteq w'$ as increase in knowledge from w to w', and the relationship $w\ R\ v$ as asserting a possible "future" evolution of knowledge from state w to state v. A covering $w \lhd \alpha$ can be understood as defining a "locality" of knowledge states α capturing "choices" in knowledge local to w. As we will see, the covering relation allows us to model the truth of a disjunctive formula $A \vee B$ at a world w by considering the truth of choices A and B in some locality α that covers w.

Before stating the conditions imposed on a cover system, we need two definitions to ease their presentation. First, we define a relation $\preceq\ \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ as $\alpha \preceq \alpha'$ iff for all worlds $v' \in \alpha'$ there exists a world $v \in \alpha$ such that $v \sqsubseteq v'$. Second, we define an operator $[R] : \mathcal{P}(W) \to \mathcal{P}(W)$ as $[R]X = \{w \mid \forall w', v.\, w \sqsubseteq w' \text{ and } w'\ R\ v \text{ implies } v \in X\}$. Below are the conditions on cover systems, with the last two aimed at modelling the modal fragment of $\mathsf{FCK}_\Box$.

- *Local Refinability*: If $w' \sqsupseteq w \lhd \alpha$, then $\exists \alpha'.\, w' \lhd \alpha' \succeq \alpha$.
- *Local Inclusion*: If $w \lhd \alpha$, then $\{w\} \preceq \alpha$
- *Local Identity*: $w \lhd \{w\}$
- *Local Transitivity*: If $w \lhd \alpha$ and $(\forall v \in \alpha.\, \exists \beta_v . v \lhd \beta_v)$, then $w \lhd \bigcup_{v \in \alpha} \beta_v$
- *Modal Localization*: If $w \lhd \alpha \subseteq [R]X$, $w \sqsubseteq w'$ and $w'\ R\ v$, then $\exists \beta.\, v \lhd \beta \subseteq X$.
- *Lock Refinability*: If $w\ R\ v$ and $v \sqsubseteq v'$, then $\exists w'.\, w \sqsubseteq w'\ R\ v'$.

We say that a subset $X \subseteq W$ is an *up-set* when it satisfies the condition that if $w \sqsubseteq w'$ and $w \in X$ then $w' \in X$, *localized* when $\exists \alpha.\, w \lhd \alpha \subseteq X$ implies $w \in X$, and a *localized up-set* when it satisfies both conditions. The above conditions respectively ensure that *truth sets* of formulas are localized and truth sets of contexts are up-sets, as we will see in Lemma 1, and allow us to prove soundness of cover semantics for judgments in $\mathsf{FCK}_\Box$.

A *cover model* $\mathcal{M} = (C, V)$ for $\mathsf{FCK}_\Box$ couples a cover system C with a valuation function V mapping propositional atoms to localized up-sets of W. This means, for an atom p, $V(p)$ must be an up-set of the preorder $(W, \sqsubseteq)$, and that if p is "locally true" at w, i.e. at all worlds in some cover α of w, then it must be true at w. Given a cover model $\mathcal{M}$ and a world w, the truth of a formula A is defined using the *satisfaction* relation $\mathcal{M}, w \Vdash A$, and extended to contexts via the relation $\mathcal{M}, w \Vdash \Gamma$ as shown below. Note the backward-looking interpretation of 🔒.

$\mathcal{M}, w \Vdash p$	iff	$w \in V(p)$			
$\mathcal{M}, w \Vdash \top$	iff	true			
$\mathcal{M}, w \Vdash \bot$	iff	$w \lhd \emptyset$	$\mathcal{M}, w \Vdash \cdot$	iff	true
$\mathcal{M}, w \Vdash A \wedge B$	iff	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$	$\mathcal{M}, w \Vdash \Gamma, A$	iff	$\mathcal{M}, w \Vdash \Gamma$ and $\mathcal{M}, w \Vdash A$
$\mathcal{M}, w \Vdash A \vee B$	iff	$\exists \alpha.\, w \lhd \alpha$ and $\forall v \in \alpha.\, \mathcal{M}, v \Vdash A$ or $\mathcal{M}, v \Vdash B$	$\mathcal{M}, w \Vdash \Gamma, 🔒$	iff	$\exists u.\, u\ R\ w$ and $\mathcal{M}, u \Vdash \Gamma$
$\mathcal{M}, w \Vdash A \Rightarrow B$	iff	$\forall w'.\, w \sqsubseteq w'$ and $\mathcal{M}, w' \Vdash A$ implies $\mathcal{M}, w' \Vdash B$			
$\mathcal{M}, w \Vdash \Box A$	iff	$\forall w', v.\, w \sqsubseteq w'$ and $w'\ R\ v$ implies $\mathcal{M}, v \Vdash A$			

Lemma 1 (Key Lemma)**.** *For any formula A, the truth set $|A|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash A\}$ is a localized up-set for an arbitrary cover model $\mathcal{M}$ of $\mathsf{FCK}_\Box$. Similarly, for any context Γ, the truth set $|\Gamma|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \Vdash \Gamma\}$ is an up-set (that need not be localized).*

Proof. By induction on formula A and context Γ. We use Modal Localization to show that truth sets of modal formulas $\Box A$ are localized, and Lock Refinability to show that truth sets of "locked" contexts $\Gamma, \text{🔒}$ are up-sets. $\square$

Key insight Cover semantics for $\mathsf{FCK}_\Box$ gives us an elegant semantic characterisation of the seemingly ad hoc modification to the ($\bot$-E) and ($\vee$-E) rules. In an attempt to prove completeness by constructing a canonical model (with contexts as worlds) for $\mathsf{FCK}_\Box$, there is a tension in satisfying the Modal Localization rule that is relieved by modifying the ($\bot$-E) and ($\vee$-E) rules to allow an arbitrary Γ' as above. We construct the canonical model by taking contexts for worlds, defining the accessibility relation as $\Gamma \; R \; \Delta$ iff $\exists \Gamma'. \, \Delta = \Gamma, \text{🔒}, \Gamma'$ and $\text{🔒} \notin \Gamma'$, and the covering relation as:

$$\Gamma \lhd \{\Gamma\} \qquad\qquad \frac{\Gamma \vdash \bot}{\Gamma, \Gamma' \lhd \emptyset} \qquad\qquad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A, \Gamma' \lhd \alpha \qquad \Gamma, B, \Gamma' \lhd \beta}{\Gamma, \Gamma' \lhd \alpha \cup \beta}$$

Consider as an example the locked context $A \vee B, \text{🔒}$. Using the above definition of the covering relation, we can show that $A \vee B, \text{🔒}$ is covered by the set consisting of the contexts $A \vee B, A, \text{🔒}$ and $A \vee B, B, \text{🔒}$ as follows.

$$\frac{A \vee B \vdash A \vee B \qquad A \vee B, A, \text{🔒} \lhd \{(A \vee B, A, \text{🔒})\} \qquad A \vee B, B, \text{🔒} \lhd \{(A \vee B, B, \text{🔒})\}}{A \vee B, \text{🔒} \lhd \{(A \vee B, A, \text{🔒}), (A \vee B, B, \text{🔒})\}}$$

Intuitively, the covering relation allows us to "branch" over the truth of the formula $A \vee B$, akin to the ($\vee$-E) rule, and define a locality with contexts where the disjuncts A and B are readily available.

In contrast to the covering relation used for IPL, which omits the arbitrary context Γ', our definition crucially mimics the modified rules to allow such a Γ'. In the absence of this modification, Modal Localization fails to hold. A similar link holds between the modification of the (Hyp) rule and satisfaction of the Lock Refinability condition.

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