Higher-order Kripke models for intuitionistic and other non-classical modal logics

Abstract

In this talk I will discuss higher-order ("nested") Kripke models [1], a generalization of traditional Kripke models that is remarkably close to Kripke's original idea – both mathematically and conceptually. Standard models are now 0-ary models, whereas n-ary models for $n > 0$ are models whose set of objects ("possible worlds") contain only $(n-1)$-ary models. The models for non-classical logics are introduced after the paradigmatic cases of intuitionistic modal logics IK and MK (a logic stronger than IK) are considered. The n-ary models (for $n > 0$) define a concept of "alternative" for any substantial interpretation of the $(n-1)$-ary models.

The key ideas behind the new semantics are the nesting of models, the use of particular worlds w as "designated points" for their own modal definitions, and the use of relations between models at each level n.

We start by defining higher-order models as follows:

Definition 1 *Any Kripke model is a* 0*-ary model* $\langle M, \succ, v\rangle$*, where* W *is a set of objects,* R *a binary relation over* W*, and* v *a function assigning sets of atoms to elements of* W*.*

Definition 2 *A* n*-ary model is a Kripke model* $\langle M, \succ, v\rangle$*, where* W *is a set of* $(n-1)$*-ary models.*

In the case of IK and MK, 0-ary models are defined as the traditional intuitionistic propositional Kripke models [2, pg. 21], and modal models are defined using only 1-ary models. In order to notationally distinguish between 1-ary and 0-ary models in this abstract, we denote 1-ary models by $\langle M, \succ, f\rangle$ and 0-ary models K by $\langle W_K, \leq_K, v_k\rangle$. We then move from the abstract definition of model to a concrete intuitionistic one as follows:

Definition 3 *A set of* 0*-ary models is homogeneous if its models are defined over the same frame (e.g same sets* W *and relation* $\leq$*), and partially homogeneous if it contains a model* M *such that every other model is defined over a generated subframe [3, pg. 28] of the frame of* M*.*

Definition 4 *An* 1*-ary model is a model for* IK *if its set* M *is partially homogeneous, and a model for* MK *if it is homogeneous.*

When we consider n-ary models in general, in any model $\langle M, \succ, v\rangle$ of level $n > 0$, validity of $\Box A$ or $\Diamond A$ in a world $w \in M$ (which is a $(n-1)$-ary model) depends only on which formulas are valid in other models w' of M such that $w \succ w'$ whose frames coincide at least in part with that of w. In the case of 1-ary models for IK, this is achieved through the following clauses, where 1 is a 1-ary model, K one of its 0-ary models, and $w \in W_K$:

$\vDash^{1,K}_{w} \Box A \iff$ for all w' with $w \leq_K w'$, if $K \succ K'$ and $w' \in W_{K'}$ then $\vDash^{1,K'}_{w'} A$;

$\vDash^{1,K}_{w} \Diamond A \iff$ there is a K' with $K \succ K'$ such that $w \in K'$ and $\vDash^{1,K'}_{w} A$;

A modal accessibility $K \succ K'$ is vacuous whenever the intersection of the objects in the frames of K and K' is empty. If the intersection is non-empty, something is necessary in a world w iff it is true in all occurrences of the same w or in the $w \leq w'$ in other accessible models, and it is possible if it is true in the w of at least one accessible model.

Intuitively, given any substantial interpretation of w and M, the definitions consider only alternative versions of the same entities; for instance, if w is interpreted as a particular moment in time and M as a timeline, modal validity in w is defined by considering only what is or is not true in the same moment w in time (or future moments) but in the accessible "alternative timelines". The traditional interpretation of accessibility relations in Kripke models as establishing different "possible worlds" is now lifted to a higher level and applied to Kripke models themselves by considering differences in the interpretation of their shared structure.

This perspective becomes clearer when we consider clauses for MK, which is the strongest possible intuitionistic modal logic obtainable by requiring birelational models to satisfy frame conditions [2, pg. 46][4]. The clauses only considers alternative versions of the same world:

$\vDash_w^{1,K} \Box A \iff$ for all K' such that $K \succ K'$, we have $\vDash_w^{1,K'} A$;

$\vDash_w^{1,K} \Diamond A \iff$ there is a K' with $K \succ K'$ such that $\vDash_w^{1,K'} A$;

The only intuitionistic thing about semantics for MK is the use of intuitionistic 0-ary models, so if we were to use 0-ary models for some other logic we would obtain a (very strong) modal logic for it. Weaker logics can also be obtained, although in some cases (such as that of IK) the semantic clauses might require reference to the specific non-modal relations defined for objects of the models. It is also the case that imposing conditions on n-ary relations for any modal logic yields the semantics traditionally obtained through the same conditions (e.g. by requiring the 1-ary relation to be reflexive and transitive the semantics for IK becomes one for $IS4$).

We are currently working with collaborators to show that, if models are not required to share any structure, the framework provides a semantics for the intuitionistic modal logic WK [5], and a simple modification yields one for CK [4]. We are also investigating new sequent calculi obtainable by reflecting the semantic structures of the new models. The relationship between the semantics and first-order intuitionistic structures is yet to be investigated. Our results indirectly show that a mapping into first-order intuitionistic logic is available by mapping 1-ary models into birelational models, but it is unclear what a "direct mapping" would look like.

Although 1-ary validity in models for IK and MK is fully determined by validity in 0-ary models (in the sense that something is valid in a 1-ary model iff it is valid in all its 0-ary models), this needs not always be the case. The framework may be capable of providing Kripke-like semantics to Kripke-incomplete logics if n-ary validity is defined in a way that makes it irreducible to $(n-1)$-ary validity. This might open the door to simple but conceptually interesting characterisations of an ever broader class of intuitionistic modal logics, including non-normal ones.

Keywords: Intuitionistic Logic, Intuitionistic Semantics, Modal Semantics, Relational Semantics, Higher-order Kripke models.

References

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