Second-order Intuitionistic Tense Logic

Justus Becker Anupam Das Sonia Marin Paaras Padhiar

Second-order intuitionistic propositional logic ($\mathsf{IPL2}$) extends propositional logic with quantification over propositions. Notably, even in the intuitionistic setting, positive connectives can be encoded with negative connectives – e.g. $A \vee B$ is equivalent to $\forall X((A \to X) \to (B \to X) \to X)$, $\bot$ is equivalent to $\forall X X$. This is contrary to intuitionistic first-order logic where the standard connectives are independent.

Intuitionistic tense logic (IKt) was first studied by Ewald [6]. In IKt, the box modality $\Box$ and the 'backwards' modality $\blacksquare$ are not De Morgan dual to their diamond counterparts $\Diamond$ and $\blacklozenge$. Whilst this was developed independently, IKt follows the 'intuitionistic' discipline introduced by Fischer Servi [7, 8], Plotkin and Stirling [11] and Simpson [13]. Other intuitionistic variants of modal logic with diamonds have been studied with no common consensus on the correct discipline, e.g. [12, 17, 3, 5, 10].

In this abstract, we define a *second-order variant of intuitionistic tense logic* ($\mathsf{IKt2}$) and summarise the results given in [1]. Remarkably, with the power of second-order quantification, we can *encode* the diamonds in terms of the negative fragment of the language:

$$\Diamond A := \forall X(\Box(A \to \blacksquare X) \to X) \qquad\qquad \blacklozenge A := \forall X(\blacksquare(A \to \Box X) \to X)$$

We provide an axiomatisation, semantic foundations through birelational and predicate structures and a labelled cut-free proof system, and show their equivalence following this grand tour.

$$\begin{tikzcd}
\mathsf{IKt2} \vdash A \arrow[r, "\text{Soundness}"'] & \forall \mathfrak{B} \vDash_{\mathfrak{B}} A \arrow[r, hook, "\text{Model embedding}"'] & \forall \mathfrak{P} \vDash_{\mathfrak{P}} A \arrow[d, "\text{Countermodel}"] \\
\ell\mathsf{IKt2} \vdash A \arrow[u, "\text{Formula interpretation}"] & & \mathsf{m}\ell\mathsf{IKt2} \vdash A \arrow[ll, "\text{Proof transformation}"']
\end{tikzcd}$$

Figure 1: Grand tour of results

Axiomatisation. The language of $\mathsf{IKt2}$ is given by the grammar:

$$A ::= P \in \mathsf{Pr} \mid X \in \mathsf{Var} \mid A \to A \mid \Box A \mid \blacksquare A \mid \forall X A,$$

where P ranges over *propositions* and X ranges over *free variables*. We assume formulas of the language are *closed*. The symbol $\bot$ and connectives $\wedge$ and $\vee$ are defined in the standard way in second-order logic, with $\Diamond$ and $\blacklozenge$ as explained above.

$\mathsf{IKt2}$ is an extension of second-order propositional intuitionistic logic ($\mathsf{IPL2}$) with the axioms:

$$\mathsf{D}_\Box : \Box(A \to B) \to \Box A \to \Box B \qquad \mathsf{D}_\Box : \blacksquare(A \to B) \to \blacksquare A \to \blacksquare B \qquad \mathsf{A}_{\blacksquare\Diamond} : A \to \blacksquare\Diamond A \qquad \mathsf{A}_{\Box\blacklozenge} : A \to \Box\blacklozenge A$$

and inference rules:

$$\mathsf{mp}\,\frac{A \to B \quad A}{B} \qquad \mathsf{gen}\,\frac{A[P/X]}{\forall X A}\,P\text{ fresh} \qquad \mathsf{nec}_\Box\,\frac{A}{\Box A} \qquad \mathsf{nec}_\blacksquare\,\frac{A}{\blacksquare A}$$

Unlike in first-order [9, 4] and second-order modal logics [2], the *Barcan* formula $\forall X \Box A \to \Box \forall X A$ is not required for a complete axiomatisation but is derivable.

Birelational semantics. We define *birelational semantics* similarly to the semantics of intuitionistic tense logic [6]. To account for second-order quantifiers, we must further include a domain of sets to evaluate propositions. This restriction yields a Henkin semantics for our second-order quantifiers.

A *(second-order) birelational structure* $\mathfrak{B}$ is a tuple $(W, \leq, \mathcal{W}, R_{\mathfrak{B}})$ where: W is a set of worlds; $\leq$ is a pre-order on W; $\mathcal{W} \subseteq \mathcal{P}(W)$ is a class of *sets* (or *predicates*) that are upwards-closed (i.e. if $V \in \mathcal{W}$, $v \in V$ and $v \leq v'$, then $v' \in V$) and contains an interpretation $P_{\mathfrak{B}} \in \mathcal{W}$ for each $P \in \mathsf{Pr}$; and $R_{\mathfrak{B}} \subseteq W \times W$ is an *accessibility relation.* We furthermore require in $\mathfrak{B}$ that $R_{\mathfrak{B}}$ is a *bisimulation* on $\leq$, i.e.:

$$\forall v, w, w' \in W\,(vR_{\mathfrak{B}}w \leq w' \implies \exists v' \geq v\, v'R_{\mathfrak{B}}w') \text{ and } \forall v, v', w \in W\,(v' \geq vR_{\mathfrak{B}}w \implies \exists w' \geq w\, v'R_{\mathfrak{B}}w')$$

We expand the language by including each $V \in \mathcal{W}$ as a propositional symbol, setting $V_{\mathfrak{B}} := V$. The judgement $v \vDash_{\mathfrak{B}} A$, for $v \in W$, is defined by induction on (closed formula) A:

$$\begin{array}{lll}
v \vDash_{\mathfrak{B}} P & \text{if} & v \in P_{\mathfrak{B}} \\
v \vDash_{\mathfrak{B}} A \to B & \text{if} & \forall v' \geq v\,(v' \vDash_{\mathfrak{B}} A \implies v' \vDash_{\mathfrak{B}} B) \\
v \vDash_{\mathfrak{B}} \Box A & \text{if} & \forall v' \geq v\, \forall w' \in W\,(v'R_{\mathfrak{B}}w' \implies w' \vDash_{\mathfrak{B}} A) \\
v \vDash_{\mathfrak{B}} \blacksquare A & \text{if} & \forall v' \geq v\, \forall u' \in W\,(u'R_{\mathfrak{B}}v' \implies u' \vDash_{\mathfrak{B}} A) \\
v \vDash_{\mathfrak{B}} \forall X A & \text{if} & \forall v' \geq v\, \forall V \in \mathcal{W}\; v' \vDash_{\mathfrak{B}} A[V/X]
\end{array}$$

We write simply $\vDash_{\mathfrak{B}} A$ if $w \vDash_{\mathfrak{B}} A$ for every $w \in W$.

$\mathfrak{B}$ is *comprehensive* if, for any closed formula C (of the expanded language), it has a set $[C] = \{w \in W \mid w \vDash_{\mathfrak{B}} C\}$ in $\mathcal{W}$. A *birelational model* is a comprehensive birelational structure.

Predicate semantics. We also consider intuitionistic predicate structures. A *(second-order) predicate structure* $\mathfrak{P}$ is a tuple, $(\Omega, \leq, W, \mathcal{W}, \{P^a\}_{a \in \Omega, P \in \mathcal{W}}, \{R^a\}_{a \in \Omega})$ where: Ω is a set of *intuitionistic worlds* (or *states*); $\leq$ is a partial order on Ω; W is a set of *(modal) worlds*; $\mathcal{W} \supseteq \mathsf{Pr}$ is a set of *(modal) sets* (or *predicates*); for $P \in \mathcal{W}$ and $a \in \Omega$, there is an interpretation $P^a \subseteq W$ such that if $a \leq b$, then $P^a \subseteq P^b$; and for $a \in \Omega$, $R^a \subseteq W \times W$ is an *accessibility relation* at a, such that if $a \leq b$, then $R^a \subseteq R^b$.

We similarly expand the language of formulas by including each $V \in \mathcal{W}$ as a propositional symbol. The judgement $a, v \vDash_{\mathfrak{P}} A$, for $a \in \Omega$ and $v \in W$, is defined by induction on A:

$$\begin{array}{lll}
a, v \vDash_{\mathfrak{P}} P & \text{if} & v \in P^a \\
a, v \vDash_{\mathfrak{P}} A \to B & \text{if} & \forall b \geq a\,(b, v \vDash_{\mathfrak{P}} A \implies b, v \vDash_{\mathfrak{P}} B) \\
a, v \vDash_{\mathfrak{P}} \Box A & \text{if} & \forall b \geq a\, \forall w \in W\,(vR^b w \implies b, w \vDash_{\mathfrak{P}} A) \\
a, v \vDash_{\mathfrak{P}} \blacksquare A & \text{if} & \forall b \geq a\, \forall u \in W\,(uR^b v \implies b, u \vDash_{\mathfrak{P}} A) \\
a, v \vDash_{\mathfrak{P}} \forall X A & \text{if} & \forall b \geq a\, \forall P \in \mathcal{W}\; b, v \vDash_{\mathfrak{P}} A[P/X]
\end{array}$$

We write simply $\vDash_{\mathfrak{P}} A$ if $a, v \vDash_{\mathfrak{P}} A$ for every $a \in \Omega, v \in W$.

$\mathfrak{P}$ is *comprehensive* if, for each formula C (of the expanded language), it has a set $[C] \in \mathcal{W}$ with $[C]^a = \{w \in W \mid a, w \vDash_{\mathfrak{P}} C\}$. A *predicate model* is a comprehensive predicate structure.

Labelled proof systems. We consider a *labelled* proof system for $\mathsf{IKt2}$ adapting Simpson's calculus for IK [13] to tense modalities and second-order quantifiers [15]. Fix a set Wl of *world symbols*, written u, v, w etc. A *relational atom* is an expression vRw, where $v, w \in \mathsf{Wl}$. A *labelled formula* is an expression $v : A$ where $v \in \mathsf{Wl}$ and A is a formula. A *labelled sequent* is an expression $\mathbf{R} \,|\, \Gamma \Rightarrow \Delta$ where $\mathbf{R}$, the *relational context*, is a set of relational atoms and Γ, Δ are multisets of labelled formulas, respectively the *antecedent* and *succedent*.

The system $\mathsf{m}\ell\mathsf{IKt2}$ is given by the rules in Fig. 2. where every upper sequent of a right-rule must have a succedent of size at most one. The system $\ell\mathsf{IKt2}$ has the same rules as $\mathsf{m}\ell\mathsf{IKt2}$ with the restriction that all sequents must have succedent of size at most one.

Summary of results. To link the structures introduced above we prove that the following are equivalent (as illustrated in Fig. 1):

1. $\mathsf{IKt2} \vdash A$;
2. $\vDash_{\mathfrak{B}} A$ for all birelational models $\mathfrak{B}$;
3. $\vDash_{\mathfrak{P}} A$ for all predicate models $\mathfrak{P}$;
4. $\mathsf{m}\ell\mathsf{IKt2} \vdash \cdot \,|\, \cdot \Rightarrow w : A$ for a world symbol w;
5. $\ell\mathsf{IKt2} \vdash \cdot \,|\, \cdot \Rightarrow w : A$ for a world symbol w.

We give a brief proof sketch; for full details, the reader is referred to [1].

1. $\implies$ 2.: standard soundness argument by induction on the proof of $\mathsf{IKt2} \vdash A$.

2. $\implies$ 3.: we can view predicate models as birelational structures by essentially taking the product $\Omega \times W$ for the set of worlds of a birelational model. By induction on A we show that this embedding is faithful.

Identity:

$$\text{id}\,\frac{}{\mathbf{R} \mid v : A \Rightarrow v : A}$$

Strutural rules:

$$\mathsf{w}_l\,\frac{\mathbf{R} \mid \Gamma \Rightarrow \Delta}{\mathbf{R} \mid \Gamma, v : A \Rightarrow \Delta} \qquad \mathsf{w}_r\,\frac{\mathbf{R} \mid \Gamma \Rightarrow \Delta}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : A}$$

$$\mathsf{c}_l\,\frac{\mathbf{R} \mid \Gamma, v : A, v : A \Rightarrow \Delta}{\mathbf{R} \mid \Gamma, v : A \Rightarrow \Delta} \qquad \mathsf{c}_r\,\frac{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : A, v : A}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : A}$$

Logical rules:

$$\to_l\,\frac{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : A \quad \mathbf{R} \mid \Gamma', v : B \Rightarrow \Delta'}{\mathbf{R} \mid \Gamma, \Gamma', v : A \to B \Rightarrow \Delta, \Delta'} \qquad \to_r\,\frac{\mathbf{R} \mid \Gamma, v : A \Rightarrow v : B}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : A \to B}$$

$$\forall_l\,\frac{\mathbf{R} \mid \Gamma, v : A[C/X] \Rightarrow \Delta}{\mathbf{R} \mid \Gamma, v : \forall X A \Rightarrow \Delta} \qquad \forall_r\,\frac{\mathbf{R} \mid \Gamma \Rightarrow v : A[P/X]}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : \forall X A}\,P \text{ fresh}$$

$$\Box_l\,\frac{\mathbf{R}, vRw \mid \Gamma, w : A \Rightarrow \Delta}{\mathbf{R}, vRw \mid \Gamma, v : \Box A \Rightarrow \Delta} \qquad \Box_r\,\frac{\mathbf{R}, vRw \mid \Gamma \Rightarrow \Delta, w : A}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : \Box A}\,w \text{ fresh}$$

$$\blacksquare_l\,\frac{\mathbf{R}, uRv \mid \Gamma, u : A \Rightarrow \Delta}{\mathbf{R}, uRv \mid \Gamma, v : \blacksquare A \Rightarrow \Delta} \qquad \blacksquare_r\,\frac{\mathbf{R}, uRv \mid \Gamma \Rightarrow \Delta, u : A}{\mathbf{R} \mid \Gamma \Rightarrow \Delta, v : \blacksquare A}\,u \text{ fresh}$$

Figure 2: Rules of mℓIKt2 and ℓIKt2. A symbol is *fresh* if it does not occur in the lower sequent.

3. $\implies$ 4.: cut-free completeness of the calculus mℓIKt2 is shown by a countermodel construction. Informally, this is done by iterating two proof search phases: (i) a transfinite, invertible and non-branching phase, and (ii) a single-step, infinitely branching and non-invertible phase. The transfiniteness of proof search is due to the comprehension rule $\forall_l$ where C can be an arbitrary formula. From this transfinite tree, we can read out a predicate counter-model with non-invertible phases corresponding to intuitionistic worlds, while labels and relational atoms correspond to modal worlds and their modal relations. Finally, in order to define an appropriate domain of sets, we exploit techniques by Prawitz and Tait for completing semivaluations, à la Schütte.

4. $\implies$ 5.: by induction on the mℓIKt2 proof of $\cdot \mid \cdot \Rightarrow w : A$. Due to the absence of native positive connectives, namely $\vee$, this is much easier than the first-order and propositional cases [16].

5. $\implies$ 1.: we define a formula interpretation of labelled sequents, and we show that rules are sound in IKt2 through the interpretation similarly to [14]. This relies on the fact that $\mathbf{R}$ is a rooted (poly-)tree whose vertices we can interpret in the modal language by tense modalities.

References

[1] Justus Becker, Anupam Das, Sonia Marin, and Paaras Padhiar. The proof theory and semantics of second-order (intuitionistic) tense logic. arXiv:2602.06253, 2026.

[2] Francesco Belardinelli, Wiebe van der Hoek, and Louwe B. Kuijer. Second-order propositional modal logic: Expressiveness and completeness results. *Artificial Intelligence*, 263:3–45, October 2018.

[3] Gianluigi Bellin, Valeria de Paiva, and Eike Ritter. Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic. In *Proceedings for Methods for Modalities 2*, Amsterdam, Netherlands, November 2001.

[4] Torben Braüner and Silvio Ghilardi. First-Order Modal Logic. In *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, pages 549–620. Elsevier, first edition, 2007.

[5] Anupam Das and Sonia Marin. On Intuitionistic Diamonds (and Lack Thereof). In Revantha Ramanayake and Josef Urban, editors, *Automated Reasoning with Analytic Tableaux and Related*

Methods, volume 14278 of *Lecture Notes in Computer Science*, pages 283–301, Prague, Czech Republic, September 2023. Springer Nature Switzerland.

[6] W. B. Ewald. Intuitionistic tense and modal logic. *Journal of Symbolic Logic*, 51(1):166–179, March 1986.

[7] Gisèle Fischer Servi. On Modal Logic with an Intuitionistic Base. *Studia Logica*, 36(3):141–149, 1977.

[8] Gisèle Fischer Servi. Axiomatizations for some intuitionistic modal logics. *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, 42(3):179–194, 1984.

[9] Melvin Fitting and Richard L. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers, Dordrecht, Netherland, 1998.

[10] Jim de Groot, Ian Shillito, and Ranald Clouston. Semantical Analysis of Intuitionistic Modal Logics between CK and IK. In *2025 40th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 169–182, Singapore, June 2025. IEEE.

[11] Gordon Plotkin and Colin Stirling. A framework for Intuitionistic Modal Logics: Extended Abstract. In Joseph Y. Halpern, editor, *Proceedings of the 1986 Conference on Theoretical Aspects of Reasoning about Knowledge*, TARK '86, pages 399–406, Monterey, California, United States, March 1986. Morgan Kaufmann Publishers Inc.

[12] Dag Prawitz. *Natural Deduction: A Proof-Theoretical Study*. PhD thesis, Stockholm University, 1965.

[13] Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, July 1994.

[14] Lutz Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In Frank Pfenning, editor, *Foundations of Software Science and Computation Structures*, pages 209–224. Springer Berlin Heidelberg, 2013.

[15] Gaisi Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 2nd edition, 1987.

[16] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Number 43 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2nd edition, July 2000.

[17] Duminda Wijesekera. Constructive modal logics I. *Annals of Pure and Applied Logic*, 50(3):271–301, December 1990.