Equality of Proofs in Substructural Intuitionistic Modal Logics

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Abstract

We develop non-commutative linear intuitionistic versions of modal logics K, T, K4 and S4 (that we call SIK, SIT, SIK4 and SIS4) with multiplicative truth, conjunction and box (and also implications) in terms of sequent calculi. Each of these sequent calculi has cut admissible and is sound and complete, in particular in regards to equality of proofs, wrt. the appropriate categorical semantics. The box modality is interpreted as a strong monoidal endofunctor in the case of SIK, and progressively strengthens to a strong monoidal comonad in the case of SIS4. We use a sequent format where antecedents are lists of formulae indexed by natural numbers for box-depths. For a variant SIS4* of SIS4 where box is an idempotent strong monoidal comonad (satisfying more equations than in the basic non-idempotent case), these indices can be restricted to 0, 1. This corresponds to unboxed and boxed compartments of antecedents in sequent calculi for logics with exchange.

This paper grew out of our wish to understand the design space for intuitionistic modal logics justifiable by sensible categorical semantics and enjoying good structural proof theory. We are specifically interested in formalisms of not just provability but of proofs coming with a notion of proof equality that should also arise from the semantics. To separate concerns, we begin with a substructural setting with no exchange, weakening or contraction and with a smallest meaningful selection of connectives, namely only multiplicative truth and conjunction together with box (tangentially we also discuss implications). The idea is to be able to see everything clearly, without interferences of different features, in this minimalistic setting and to add the structural rules and further connectives later, i.e., to follow the same approach as in our earlier project on skew substructural intuitionistic logic [6].

Our intended semantics has as a model a monoidal category with an endofunctor on it (interpreting box), justifying inference rule $A \to B \;/\; \Box A \to \Box B$, with some structure on the endofunctor. Specifically, we require the endofunctor to be strong monoidal, corresponding to axioms $\mathsf{I} \cong \Box\mathsf{I}$ and $\Box A \otimes \Box B \cong \Box(A \otimes B)$, for the basic system SIK. Optionally, it may come with additional data and equations that make it a strong monoidal copointed functor, semicomonad or comonad, with axioms $\Box A \to A$ or/and $\Box A \to \Box\Box A$, for systems SIT, SIK4, SIS4.

The free monoidal category with a strong monoidal endofunctor, copointed endofunctor, semicomonad or comonad can be straightforwardly described as a Hilbert-like system with an equational theory stipulating which proofs we consider equal, corresponding to the equations of monoidal category, strong monoidal endofunctor etc.

But we are interested in sequent calculus counterparts for these Hilbert systems. Our design is based on sequents of the form $\Gamma \to A$ where Γ is a list of natural-number indexed formulae and A is a single formula. An indexed formula A^i intuitively stands for $\Box^i A$ and a list $A_1^{i_1}, \ldots, A_n^{i_n}$ stands for $\Box^{i_1} A_1 \otimes (\ldots \otimes (\Box^{i_n} A_n \otimes \mathsf{I})) \ldots)$, so the index on a formula in the antecedent of a sequent records its box-depth, i.e., the number of boxes that were decomposed to reach it (under the root-first, i.e., backward, reading of the proof).

The following are the inference rules of SIK.

$$\frac{}{A \to A}\ ax \qquad \frac{}{\to \mathsf{I}}\ \mathsf{I}R \quad \frac{\Delta, \Delta' \to C}{\Delta, \mathsf{I}^i, \Delta' \to C}\ \mathsf{I}L^i \qquad \frac{\Gamma_0 \to A \quad \Gamma_1 \to B}{\Gamma_0, \Gamma_1 \to A \otimes B}\ \otimes R \quad \frac{\Delta, A^i, B^i, \Delta' \to C}{\Delta, A \otimes B^i, \Delta' \to C}\ \otimes L^i$$

$$\frac{\Gamma \to A}{\Gamma^{1+} \to \Box A}\ \Box R \qquad \frac{\Delta, A^{i+1}\Delta' \to C}{\Delta, \Box A^i, \Delta' \to C}\ \Box L^i$$

(Γ^{i+} is the list of formulae obtained from Γ by incrementing all indices by i.)

The systems SIT, SIK4 and SIS4 additionally have the first, the second or both of the following rules of structural character:

$$\frac{\Delta, A^{i+j}, \Delta' \to C}{\Delta, A^{i+1+j}, \Delta' \to C}\ T^{i,j} \qquad \frac{\Delta, A^{i+2+j}, \Delta' \to C}{\Delta, A^{i+1+j}, \Delta' \to C}\ 4^{i,j}$$

In all these systems, the following cut rule is admissible:

$$\frac{\Gamma \to A \quad \Delta, A^i, \Delta' \to C}{\Delta, \Gamma^{i+}, \Delta' \to C}\ cut$$

The generating equations of the equational theories of proofs consist for all four sequent calculi of a number of η-conversions and a number of permutative conversion equations.

The Hilbert systems are equivalent to their sequent calculus counterparts in the following sense. For any Γ and A, the proofs of $\Gamma \to A$ in the sequent calculus are in an effective bijection with the proofs of $\langle \Gamma \rangle \to A$ in the Hilbert system (there are translations in both directions). Moreover the bijection (the translations) respects the proof equality relations of the Hilbert system and the sequent calculus.

Because of the equivalence of the Hilbert systems and the sequent calculi, both are sound and complete, in particular, also in the equational sense, wrt. the categorical semantics.

Strengthening the copointed endofunctor, semicomonad or comonad to a well-copointed endofunctor, "quasi-idempotent" semicomonad or idempotent comonad corresponds to identifying the two inequal proofs of $\Box\Box A \to \Box A$ (the outer or the inner box on the left can be deleted by axiom T). This semantics defines logics SIT*, SIK4* and SIS4* which have the same provability as SIT, SIK4, SIS4, but a coarser proof equality relation (identifying more proofs).

In terms of sequent calculus, this means that rules $T^{i,j}$, $T^{k,\ell}$ have to be equated for $i + j = k + \ell$ and ditto for $4^{i,j}$, $4^{k,\ell}$. Therefore, we can collapse them into rules T^{i+j} resp. 4^{i+j} only recording the sum of i,j.

The sequent calculus for SIS4* further admits a simplification where the indices are just 0, 1 and + gets replaced with $\vee$ (maximum). This makes it close to Kavvos' [4] and Miranda Perea et al.'s [5] natural deduction system and sequent calculus for intuitionistic S4 with 2-compartment antecedents for unboxed and boxed formulae.

Developing a subcalculus of normal proofs for each of our sequent calculi by treating the equational theory as a rewrite system and studying normalization, we can prove coherence theorems for our categorical semantics. Normal proofs serve as canonical representatives of each class of equal proofs. In SIK, there is at most one normal proof of any sequent. In SIT, SIK4, SIS4, there can be multiple proofs since it matters where to delete or duplicate a box in a tower of boxes. In the sequent calculi for SIT*, SIK4* and SIS4*, there is again at most one normal poof of any sequent. Adding implication to the selection of connectives leads to the possibility of multiple proofs of the same sequent already in SIK. In related work, Došen and Petrić [2, 3] have investigated coherence for monoidal categories with a lax monoidal endofunctor (with axioms $\mathsf{I} \to \Box\mathsf{I}$ and $\Box A \otimes \Box B \to \Box(A \otimes B)$) and a lax monoidal comonad, but not with sequent calculi.

We also consider substructural versions of intuitionistic modal logics featuring, in addition to box, a (possibly structural-only) past-tense diamond operator. Our sequent calculi for these logics imitate the Fitch-style natural deduction systems due to Clouston [1]. Antecedents are lists of formulae with nonpositive-integer indices for diamond-depths. The categorical semantics is in terms of monoidal categories with a lax monoidal endofunctor (interpreting box) that has a strong monoidal left adjoint (interpreting diamond).

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