Weak constructive modal logics systematised

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We owe the discovery of many intuitionistic modal logics to the conjunction of an assumption and a question. The assumption is that the modalities $\Box$ and $\Diamond$ are not intuitionistically interdefinable, just as the propositional connectives are not. The question is how to identify proper intuitionistic analogues of a classical modal logic.

The rejection of modal duality combined with the intuitionistic behaviour of the propositional connectives allows us to distinguish between many modal principles that are instead classical equivalent, opening a search space for the identification of intuitionistic modal logics which is barely limited. In addition to this, the notion of "intuitionistic analogue" is not univocally understood, and different interpretations can lead to different results. An insight of the potential breadth of the domain of intuitionistic modal logics is offered by the number of intuitionistic variants of the classical modal logic K that have been proposed in the literature. These include traditional solutions, such as Intuitionistic K $\mathsf{I.K}$ [13, 19], Wijesekera's K $\mathsf{W.K}$ [21] and Constructive K $\mathsf{C.K}$ [6, 11], as well as several more recent proposals (see for instance [3, 4, 5, 9, 10]), all of them with their own motivations and peculiarities.

Once an intuitionistic variant of a classical modal logic is proposed, an orthogonal research direction consists in defining *the same kind* of intuitionistic variants for other classical modal logics. In other words, given a classical logic L, an intuitionistic variant $\mathsf{int}(\mathsf{L})$ of it, and another classical logic L', we want to find the system x such that $\mathsf{int}(\mathsf{L}) : \mathsf{L} = x : \mathsf{L}'$. In order to determine x and avoid arbitrary choices, we first need to grasp the key aspects of that specific way of being an intuitionistic variant of a classical modal logic, and then apply them to L', if it is possible.

The most elegant and general systematisation of a family of intuitionistic modal logics has been offered for $\mathsf{I.K}$ and its extensions. Simpson [19] showed that every logic $\mathsf{I.L}$ corresponds via standard translation to the set of first-order formulas valid in the intuitionistic logic structures defined by the frame condition (understood as a FOL formula) that characterises the relational models for $\mathsf{I.L}$. A different but equivalent characterisation of the same logics has been offered in [14, Chapter 10.2], where the logics $\mathsf{I.L}$ are shown to be reducible to modal logic products $\mathsf{S4} \otimes \mathsf{L}$ via suitable extensions of Gödel's translation of IPL into $\mathsf{S4}$ [12, 22].

In the case of constructive modal logics (CMLs), this problem has mostly been addressed starting from the axiomatic systems. Constructive counterparts for the modal logics between K and $\mathsf{S5}$ have been presented in [1, 2, 17] extending $\mathsf{C.K}$ with pairs of $\Box$- and $\Diamond$-versions of their characteristic axioms, like $(T_\Box)$ $\Box A \supset A$, $(T_\Diamond)$ $A \supset \Diamond A$ and $(4_\Box)$ $\Box A \supset \Box\Box A$, $(4_\Diamond)$ $\Diamond\Diamond A \supset \Diamond A$ (with the exception of D which is taken in the usual formulation $\Box A \supset \Diamond A$). However, this approach has a limitation, due to the fact that the $\Box$- and the $\Diamond$-version of a same axiom are not always simultaneously valid. For instance, $\mathsf{C.K}$ validates the axiom $(C_\Box)$ $\Box A \wedge \Box B \supset \Box(A \wedge B)$ and the necessitation rule $A/\Box A$, but it does not validate their $\Diamond$-counterparts $(C_\Diamond)$ $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ and $(N_\Diamond)$ $\neg\Diamond\bot$. Moreover, $\Box$- and $\Diamond$-versions also exist for the axiom D, namely $(D_\Box)$ $\neg(\Box A \wedge \Box\neg A)$ and $(D_\Diamond)$ $\Diamond A \vee \Diamond\neg A$, but neither of them is derivable in $\mathsf{C.KD}$. As a consequence, it is not always clear how to extend this family of systems from a purely axiomatical point of view.

The paper [7] presented two distinct but ultimately equivalent characterisations for a family of Wijesekera-style constructive modal logics. By $\mathsf{W.K}$ we refer to the propositional fragment of Wijesekera's Constructive

Concurrent Dynamic logic [21]. This logic has very simple proof-theoretical and semantical relations with classical K. On the one hand, it can be obtained by restricting a sequent calculus for K to sequents with at most one formula in the consequent. On the other hand, modal formulas are interpreted in birelational models simply by generalising their classical interpretations to all $\leq$-successors (for instance, the classical clause for $w \Vdash \Diamond A$ here becomes $\forall v \geq w.\exists u.v\mathcal{R}u \;\&\; u \Vdash A$). The idea of [7] was to use these properties as criteria for the definition of analogous counterparts of other classical modal logics. A set of 14 classical modal logics were selected according to the following requirements: They all have a cut-free Gentzen sequent calculus available, and they have a uniform possible-world semantic characterisation. These logics are included between the monotone non-normal modal logic M, characterised only by the monotonicity rule $A \supset B / \Box A \supset \Box B$, and KT. The intuitionistic systems obtained by restricting the sequent rules to single-succedent were then compared with those obtained by generalising the modal semantic clauses. The outcome was that for each of the 14 logics, the two approaches generate the same Wijesekera-style counterpart. Moreover, the semantics suggested a uniform reduction of the logics $\mathsf{W.L}$ into modal logic *fusions* $\mathsf{S4} \oplus \mathsf{L}$, thus relating Wijesekera-style logics to a different kind of combination of classical modalities.

In this talk, I will discuss the results of the paper [8], where a similar procedure is applied to CMLs in the family of $\mathsf{C.K}$. Despite the apparent similarity of $\mathsf{W.K}$ and $\mathsf{C.K}$, the latter system has some proof-theoretical and semantical peculiarities that suggest a connection between $\mathsf{C.K}$ and modal logics with a *minimal* basis: In terms of calculus, the sequent rules of $\mathsf{C.K}$ always contain exactly one formula in the consequent. In terms of semantics, the birelational models for $\mathsf{C.K}$ contain fallible worlds needed to invalidate the axiom $\neg\Diamond\bot$. This connection with minimal logic is then suggested by the multi vs. single-succedent relations between the classical, intuitionistic and minimal G1-style sequent calculi [20], as well as by the possible-world semantics with fallible worlds of minimal logic [18]. The aim of the paper is to clarify this connection and use it as a map to organise the CMLs already studied in the literature as well as define similar counterparts for other classical modal logics.

The analysis starts from the definition of a minimal variant of K, called $\mathsf{M.K}$. This is obtained by restricting the G1-calculus of K to sequents with *exactly one* formula in the succedent. Equivalently, it is obtained by extending the minimal possible-world models with a modal binary relation and interpreting the modalities as in $\mathsf{W.K}$-models. In turn, this logic can be reduced to $\mathsf{S4} \oplus \mathsf{K}$ via a natural extensions of the Gödel-Johansson translation of MPL into $\mathsf{S4}$ [15, 16]. We then observe that $\mathsf{C.K}$ can be obtained from $\mathsf{M.K}$ in either of the following ways: (1) Proof-theoretically, by extending the G1-calculus for IPL with the modal rules of $\mathsf{M.K}$. (2) Semantically, by adding conditions on the set of fallible worlds so that they satisfy *ex falso quodlibet* $\bot \supset A$. (3) Axiomatically, by extending $\mathsf{M.K}$ with the axiom $\bot \supset A$.

Subsequently, we extend this analysis to the 14 classical logics L considered in [7]. We first define minimal variants of them which are both the single-succedent restriction of their sequent calculi and the preimages of the reduction to $\mathsf{S4} \oplus \mathsf{L}$ through the extended Gödel-Johansson translation. We then define corresponding constructive variants by applying the above relations (1), (2), (3) between $\mathsf{M.K}$ and $\mathsf{C.K}$. We obtain as a result uniform constructive variants for each of the 14 classical logics which contain the systems $\mathsf{C.K}$, $\mathsf{C.KD}$ and $\mathsf{C.KT}$ already studied in the literature as well as constructive variants of additional classical logics.

An interesting property of these logics can be observed when they are compared with the Wijesekera-style ones. While all logics $\mathsf{W.L}$ validate the principle of *modalised non-contradiction* $\neg(\Box A \wedge \Diamond\neg A)$ (which can be seen as the intuitionistically legitimate direction of the duality principle), the same does not apply to the logics $\mathsf{C.L}$, where the modalities are essentially disconnected. As a matter of fact, each logic $\mathsf{W.L}$ can be obtained by adding $\neg(\Box A \wedge \Diamond\neg A)$ to the corresponding logic $\mathsf{C.L}$.

The logics considered in the paper are all characterised by non-iterative modal axioms, thus excluding well-known modal axioms such as $4, 5, B$. In the final part of the talk, I will discuss the limitations of this approach and possible ways to overcome them.

Definitions. Let $Atm = \{p_0, p_1, p_2, \dots\}$ be a countable set of propositional variables. The language $\mathcal{L}$ is defined by the grammar $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A$, where $p \in Atm$. We also define $\top := \bot \supset \bot$ and $\neg A := A \supset \bot$. We consider the following modal axioms.

$mon_\Box \; \dfrac{A \supset B}{\Box A \supset \Box B}$	$K_\Box$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$N_\Box$	$\Box\top$	$P_\Box$	$\neg\Box\bot$
	$K_\Diamond$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$N_\Diamond$	$\neg\Diamond\bot$	$P_\Diamond$	$\Diamond\top$
$mon_\Diamond \; \dfrac{A \supset B}{\Diamond A \supset \Diamond B}$	$C_\Box$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$T_\Box$	$\Box A \supset A$	D	$\Box A \supset \Diamond A$
	$C_\Diamond$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$	$T_\Diamond$	$A \supset \Diamond A$		

Definition 1. Minimal modal logics *are axiomatically defined in the language $\mathcal{L}$ extending any axiomatisation of minimal propositional logic, formulated in $\mathcal{L}$, with the following modal axioms and rules.*

M.M := $mon_\Box$, $mon_\Diamond$	M.MD := M.M + D, $P_\Diamond$	M.MT := M.M + $T_\Box$, $T_\Diamond$
M.MN := M.M + $N_\Box$	M.MND := M.MN + D	M.MNT := M.MN + $T_\Box$, $T_\Diamond$
M.MC := M.M + $C_\Box$, $K_\Diamond$	M.MCD := M.MC + $P_\Diamond$	M.MCT := M.MC + $T_\Box$, $T_\Diamond$
M.K := M.MC + $N_\Box$	M.KD := M.K + D	M.KT := M.K + $T_\Box$, $T_\Diamond$

M.MP := M.M + $P_\Diamond$
M.MNP := M.MN + $P_\Diamond$

Constructive modal logics *are defined as* C.L := M.L + $\bot \supset A$, *and* Wijesekera-style constructive modal logics *are defined as* W.L := C.L + $\neg(\Box A \wedge \Diamond\neg A)$.

Definition 2. *A* minimal neighbourhood model *is a tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\leq$ is a reflexive and transitive binary relation on $\mathcal{W}$, $\mathbb{F} \subseteq \mathcal{W}$ is a $\leq$-upward closed set (if $w \in \mathbb{F}$ and $w \leq v$, then $v \in \mathbb{F}$) of so-called* fallible worlds, *$\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a* hereditary *valuation function (if $w \in \mathcal{V}(p)$ and $w \leq v$, then $v \in \mathcal{V}(p)$), and $\mathcal{N}$ is neighbourhood a function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ possibly satisfying some of the following conditions: (C) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$. (N) $\mathcal{N}(w) \neq \emptyset$. (D) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \neq \emptyset$. (P) $\emptyset \notin \mathcal{N}(w)$. (T) If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$. The forcing relation $w \Vdash A$ is inductively defined as follows:*

$w \Vdash p$	*iff*	$w \in \mathcal{V}(p)$;	$w \Vdash B \supset C$	*iff*	$\forall v \geq w : v \Vdash B \Rightarrow v \Vdash C$.
$w \Vdash \bot$	*iff*	$w \in \mathbb{F}$;	$w \Vdash \Box B$	*iff*	$\forall v \geq w, \exists \alpha \in \mathcal{N}(v) : \alpha \Vdash^\forall B$;
$w \Vdash B \wedge C$	*iff*	$w \Vdash B$ and $w \Vdash C$;	$w \Vdash \Diamond B$	*iff*	$\forall v \geq w, \forall \alpha \in \mathcal{N}(v) : \alpha \Vdash^\exists B$;
$w \Vdash B \vee C$	*iff*	$w \Vdash B$ or $w \Vdash C$;			

A constructive neighbourhood model *is a minimal neighbourhood model where the following hold for every $w \in \mathbb{F}$: (i) $w \in \mathcal{V}(p)$ for all $p \in Atm$. (ii) There is $\alpha \in \mathcal{N}(w)$ such that $\alpha \subseteq \mathbb{F}$. (iii) For all $\alpha \in \mathcal{N}(w)$, $\alpha \cap \mathbb{F} \neq \emptyset$.*

Summary of results.

M.L	C.L	W.L
Definition 1	M.L + $\bot \supset A$	C.L + $\neg(\Box A \wedge \Diamond\neg A)$
Minimal neighbourhood models	Constructive neighbourhood models	Minimal neighbourhood models without fallible worlds
G1 prop. and modal rules with *exactly one* formula in the succedent	G1 prop. rules with *at most one* formula in the succedent, G1 modal rules with *exactly one* formula in the succedent	G1 prop. and modal rules with *at most one* formula in the succedent

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