Towards a logical characterization of Effectful Contextual Modal Type Theory

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Background on Effectful Contextual Modal Type Theory

In 2021, N. Zyuzin and A. Nanevski [13] proposed a calculus for the theory of algebraic effects [9, 11, 12], where effectful signature is treated as a *variable context*, and effects are tracked via *indexed modalities* inherited from Contextual Modal Type Theory (CMTT) [7]. In the resulting calculus, named ECMTT (Effectful Contextual Modal Type Theory), effect information is encoded in the *modal type* $[\Psi]A$, classifying computations of type A that may invoke exactly the operations declared in the algebraic theory Ψ.

On Contextual Modal Type Theory. CMTT is a contextual variant of constructive modal S4 as defined in [8]. Its typing judgments are of the form $\Delta; \Gamma \vdash e : A$, where Γ holds ordinary *value variables* $x : A$ (intuitively, making the assumption that A is *contingently* true at the given world) and Δ holds *modal variables* $u :: A[\Psi]$, annotated with a context Ψ of variables saying that A is *valid relative to* Ψ (i.e., true in every world in which every B in Ψ is true). Two of the main rules of CMTT are the following.

$$\Box I\ \frac{\Delta; \Psi \vdash e : A}{\Delta; \Gamma \vdash \text{box } \Psi.e : [\Psi]A} \qquad \frac{\Delta; \Gamma \vdash e_1 : [\Psi]A \qquad \Delta, u :: A[\Psi]; \Gamma \vdash e_2 : B}{\Delta; \Gamma \vdash \text{let box } u = e_1 \text{ in } e_2 : B}\ \Box E$$

The *introduction form* **box** $\Psi.\ e$ binds all variables of Ψ in the scope of e, simultaneously removing the outside context Γ from e's premise (reading the rule $\Box I$ from the bottom up). The *elimination form* **let box** $u = e_1$ **in** e_2 (rule $\Box E$) binds the unboxed body of e_1 to the modal variable u; occurrences of u in e_2 must be guarded by *explicit substitutions* $u\langle\sigma\rangle$ that supply definitions for the variables in Ψ. ECMTT adopts and readapts the aforementioned concepts: contexts Ψ become algebraic theories and explicit substitutions become effect handlers [10].

Accomodating algebraic effects. Consider, as an example, the operation *get* that reads the current state and the operation *set* that updates it. An algebraic theory, e.g. the state theory

$$\mathit{St} \triangleq \mathit{get} \div \mathit{unit} \Rightarrow \mathit{int},\ \mathit{set} \div \mathit{int} \Rightarrow \mathit{unit},$$

is treated as a variable context locally bound by the **box** constructor. The term

$$\mathit{incr} \triangleq \textbf{box}\ \mathit{St}.\ x \leftarrow \mathit{get}();\ \_ \leftarrow \mathit{set}(x+1);\ \textbf{ret}\ x$$

has type $[\mathit{St}]\mathit{int}$, and the boxed term is a pure (i.e. not effectful) expression.

As mentioned above, context reachability in CMTT is witnessed by *explicit substitutions*; in ECMTT it is witnessed by *effect handlers*. Consider the following

$$\mathit{handlerSt} \triangleq (\mathit{get}(x,k,z) \to \text{cont}\ k\ z\ z,\ \mathit{set}(x,k,z) \to \text{cont}\ k\ ()\ x,\ \mathit{return}(x,z) \to \textbf{ret}(x,y)).$$

A handler such as *handlerSt* specifies how each operation of *St* is interpreted, thereby transforming a computation in theory *St* into one in the empty theory.

ECMTT uses four variable judgments ($x : A$, $u :: A[\Psi]$, operations $\mathrm{op} \div A \Rightarrow B$, continuation variables $k \rightsquigarrow A \overset{S}{\Rightarrow} B$) and five term judgments (expressions, computations, statements, handlers, handling sequences). By design, the operational semantics of ECMTT is determined solely by β-reduction, as customary in other logic-based calculi (but not in other systems for algebraic effects).

The new syntactic category of (effectful) *computations* sequences bindings of algebraic operations and continuation calls in monadic-style notation $x \leftarrow s;\ c$, and terminates with $\mathbf{ret}\ e$. The category of *expressions* remains purely functional. In contrast with the CMTT setting, the modal type $[\Psi]A$ now classifies computations, not expressions.

The context Γ (formerly holding value variables) now holds *algebraic operations* $op \div A \Rightarrow B$ and *continuation variables* $k \rightsquigarrow A \overset{S}{\Rightarrow} B$. The context Ψ (used in $[\Psi]A$) ranges only over operation-only subsets (algebraic theories). Value variables $x : A$ migrate to Δ, reflecting the fact that values are computations in the empty theory. This allows values to survive boxing, enabling parametric computations such as $incr_n \triangleq \lambda n : int.\ \mathbf{box}\ St.\ x \leftarrow get();\ y \leftarrow set(x + n);\ \mathbf{ret}\ x$.

In search for a logical interpretation of Algebraic Effects

In the case of the constructive modal logic S4, we already have a relational Kripke-style, algebraic, and categorical semantics [1, 6]; all of these are well-known and have been successfully used to study the metaproperties of the logic for which they are complete (consider, e.g., [2]). At present, however, there is no known semantic characterization of the logic defined by the ECMTT calculus, nor is there a known Hilbert-style axiomatization of it. Our task is to find such a characterization to identify for the first time a correspondence, in the style of Curry-Howard, targeting algebraic effects.[1]

The following is a list of relevant, non-interderivable theorems of ECMTT, which currently constitute an initial attempt at axiomatization within a Hilbert-style proof system.

1. $\vdash [\Psi]A \wedge [\Psi]B \to [\Psi](A \wedge B)$, for every Ψ;
2. $\vdash A \to [\Psi]A$, for every Ψ;
3. $\vdash [\Psi][\Psi]A \to [\Psi]A$, for every Ψ;
4. $\vdash [\Psi]A \to [\Psi']A$, given $\Psi \subseteq \Psi'$;
5. $\vdash A \to [\Psi]B$, whenever $\mathrm{op} \div A \Rightarrow B$ is in Ψ;
6. $\vdash [\Psi]C \to ((A_1 \to B_1) \to \cdots \to (A_n \to B_n) \to \Box C)$, where $\Psi = \{\mathrm{op}_i \div A_i \Rightarrow B_i\}_{i=1}^{n}$.

The first three theorems listed above correspond exactly to a multimodal version of propositional Lax logic [5, 4]. Following [3], we know that multimodal propositional Lax logic is sound and complete with respect to multirelational CL-Kripke models. The latter are defined as a tuple $(W, \leq, V, \{R_\Psi \mid \Psi \text{ is an algebraic signature}\}, \models)$, where W is a non-empty set of possible worlds partially ordered by $\leq$; V is a map which assigns to every propositional variable p an upset $V(p) \subseteq W$; each R_Ψ is a partial order on W; finally $\models$ is a relation between worlds and formulae recursively defined as in the intuitionistic case, where

$$w \models [\Psi]A \text{ iff forall } w' \geq w \text{ there exists } w'' \text{ such that } w' R_\Psi w'' \text{ and } w'' \models A.$$

We also require every R_Ψ to be hereditary, namely if $w \models A$ and $w R_\Psi v$, then $v \models A$.

[1] As far as the authors are aware, this would count as the first instance of a Curry-Howard-style correspondence where the computational system is developed before its logical interpretation.

It can be shown that the validity of Theorems 4, 5, and 6, in the context of the multirelational CL-Kripke models defined above, is equivalent to the following conditions imposed on the accessibility relations R_Ψ and the valuation function V.

4. $\vdash R_\Psi \subseteq R_{\Psi'}$, whenever $\Psi \subseteq \Psi'$;

5. whenever $\mathrm{op} \div A \Rightarrow B \in \Psi$ and $w \models A$, there exists $v \in W$ such that $wR_\Psi v$ and $v \models B$;

6. whenever $w \models [\Psi]C$ and $w \models A_i \to B_i$ for every $\mathrm{op} \div A_i \Rightarrow B_i \in \Psi$, there exists $v \in W$ such that $wR_\emptyset v$ and $v \models C$.

Intuitively, Theorem 4 states that whenever an algebraic signature is contained within another, it is possible to use every operation present in the first one within the second. Theorem 5, on the other hand, states that it is possible to perform $\mathrm{op} \div A \Rightarrow B$ (in the context of an appropriate algebraic signature Ψ containing the aforementioned operation) whenever an input of type A is available. Finally, Theorem 6 captures the concept of operation handling: every $\mathrm{op} \div A \Rightarrow B$ in Ψ is handled via the corresponding function of type $A \to B$.

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