Towards a simple labelled natural deduction for Constructive Modal Logics (Abstract)

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What the appropriate intuitionistic analogue of the normal modal logics should be is question without a simple answer. In the intuitionistic setting, $\Box$ and $\Diamond$ are no longer defined as dual to each other and, as a result, it is not so immediate as to what the correct rules for intuitionistic $\Box$ and $\Diamond$ should be. Most commonly, as in [5,1,9], one finds Intuitionisic Modal Logics (IMLs) defined as taking any axiomatisation of IPL and adding the following five axioms and rule of inference

$$k_1 : \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi \qquad k_3 : \Diamond\bot \to \bot$$
$$k_2 : \Box(\varphi \to \psi) \to \Diamond\varphi \to \Diamond\psi \qquad k_4 : \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi)$$
$$k_5 : (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$$
$$\frac{\vdash \varphi}{\Gamma \vdash \Box\varphi}\ \text{Nec}$$

possibly with additional axioms (such as intuitionistic variants of the T or 4 axioms). We call the logic with just the axiomatisation above, IK, as, in analogy with K, it forms the base of the IMLs. Simpson, in his thesis [9], argues that the IMLs should be considered the real intuitionistic analogue to the normal modal logics. As part of his argument for this position, Simpson shows that, using labelled formulae, a normalising natural deduction system for the IMLs arises as a result of the standard translation of such formulae into intuitionistic first-order logic. Thus, axioms k_3 to k_5 can be seen to hold in IMLs as a result of the fact that intuitionistic $\exists$ and $\forall$ also interact in this way. If, however, one restricts a sequent calculus for classical modal logic to be single conclusion, one attains a sequent calculus for an intuitionistic modal logic *without* these axioms. This begs the question: Can we develop a normalising natural deduction system with introduction and elimination rules for the modalities as in the case of IK, for this intuitionistic modal logic without k_3 to k_5? This logic, called "Constructive K" (CK), in fact, admits a non-labelled, normalising, natural deduction system, as discussed in [2]. However, this system has an unfortunate and somewhat undesirable property[1] that the modalities each have only one inference rule that act as both an introduction and an elimination rule. Not satisfied with this system, we instead propose a labelled, normalising, natural deduction system for CK, that can be considered a modification of Simpson's for IK, where $\Box$ and $\Diamond$ both have introduction and elimination rules. We do this by starting from Simpson's system for IK and, by combining the techniques of Strassburger, Das and Arisaka [1] and Masini [8], surgically removing features of the natural

[1] Undesirable, from the definitional point of view of the inference rules.

deduction system to make it sound and complete for CK. The methodology used is completely proof-theoretically motivated but certain interesting properties, obvious in the semantics, such as the downward closure condition for $\perp$ (c.f. [2,5]) come to the forefront in this calculus.

We use Strassburger, Das and Arisaka [1] as a starting point, as they have a nested sequent calculus for CK that retains cut elimination. Nested calculi [4,1] allow for strict tree-like reasoning where formulae can be considered as residing in "branches" of a tree. This tree-like reasoning is a core aspect of the proof system we develop. When eschewing ourselves of the explicitly nested structure of nested sequents, instead appealing to a labelling system, we have two choices of how to assign labels to formulae such that we preserve the tree-like reasoning structure: Firstly, one can label formulae by the sequence of nodes required to reach the "position" it is at, as used by Martini, Masini and Zorzi [7]. Whilst perfectly reasonable, doing so seems somewhat syntactically heavy and their approach to extending their system to the other constructive modal logics seems ad-hoc for each logic considered. Alternatively, one can use a simple labelling scheme, akin to that of Simpson's calculus for IK, instead demanding globally that the underlying graph of the consequence relation always remains a tree, similar to the approach taken by Goré and Ramanayake with tree-labelled sequents [6]. This approach is the approach we take, though this restriction alone is not enough. To fully capture the "nested-like" reasoning, we also need to introduce the concept of context restrictions to rules. The restriction says that the derivation of hypothesis cannot occur using open assumptions holding at certain worlds. We observe that context restrictions are not completely novel, for they are present in Masini's 2-Natural Deduction [8] for Deontic logic, albeit in a slightly different form. It turns out, these two modifications suffice to develop a normalising natural deduction system for CK. In fact, extending this calculus to the rest of the constructive modal cube is as trivial as in the case of IK with exception to the case of logics with the "B" axiom, where the approach presented breaks down since the "B" axiom seems to require cycles in the underlying graph.

To give an example of what we propose, we proceed as follows. Since our underlying consequence relation is always labelled by a tree, then we have a partial order, $\leq$, on the nodes of our tree denoting that a node is below another node. We say $x \leq y$ to mean that either $x = y$ or there exist some nodes z_i in the tree such that xRz_1 and $\ldots$ and z_nRy. If $x \leq y$, then we say that y is below x. Given a tree $\mathcal{T}$ and a set Γ containing labelled formulae, of the form φ^x, we write $\Gamma|_x = \{\varphi^y \mid \forall \varphi^y \in \Gamma,\, y \leq x\}$. We call this the restriction of Γ at x. The inference figures for the modal connectives become

$$\frac{\Gamma|_x \vdash_{\mathcal{T}} \varphi^y}{\Gamma \vdash_{\mathcal{T} \cup xRy} (\Box\varphi)^x}\ \Box_{\mathsf{I}}{}^* \qquad \frac{\Gamma|_x \vdash_{\mathcal{T}} (\Box\varphi)^x \quad \Gamma|_x \vdash_{\mathcal{T}} xRy}{\Gamma \vdash_{\mathcal{T}} \varphi^y}\ \Box_{\mathsf{E}}$$

$$\frac{\Gamma|_y \vdash_{\mathcal{T}} \varphi^y \quad \Gamma|_y \vdash_{\mathcal{T}} xRy}{\Gamma \vdash_{\mathcal{T}} \Diamond\varphi^x}\ \Diamond_{\mathsf{I}}$$

$$\frac{\Gamma|_x \vdash_{\mathcal{T}} (\Diamond\varphi)^x \quad \Gamma|_x \vdash_{\mathcal{T}} x \leq z \quad \Gamma|_x, \varphi^y \vdash_{\mathcal{T} \cup xRy} \psi^z}{\Gamma \vdash_{\mathcal{T}} \psi^z}\ \Diamond_{\mathsf{E}}{}^*$$

The rules marked with a $*$ are subject to the side condition that y appears fresh. We can now demonstrate that Simpsons proof of axiom k_5 fails in our system. In Simpsons system, we have that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\Diamond\varphi \to \Box\psi)^x]^1 \qquad \dfrac{[\varphi^y]^3 \quad [xRy]^2}{(\Diamond\varphi)^x}\,\Diamond_{\mathsf{I}}}{(\Box\psi)^x}\,\to_{\mathsf{E}} \qquad [xRy]^2}{\psi^y}\,\Box_{\mathsf{E}}}{(\varphi \to \psi)^y}\,{\to_{\mathsf{I}}}^3}{(\Box(\varphi \to \psi))^x}\,{\Box_{\mathsf{I}}}^2}{((\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi))^x}\,{\to_{\mathsf{I}}}^1$$

In our proposed system, the subproof

$$\dfrac{(\Diamond\varphi \to \Box\psi)^x \qquad \dfrac{\varphi^y \quad xRy}{(\Diamond\varphi)^x}\,\Diamond_{\mathsf{I}}}{(\Box\psi)^x}\,\to_{\mathsf{E}}$$

doesn't hold as $x \leq y$ in $\mathcal{T} \cup xRy$ so, letting $\Gamma = \{(\Diamond\varphi \to \psi)^x, \psi^y\}$, we see that $\Gamma|_x \nvdash_{\mathcal{T}\cup xRy} (\Box\psi)^x$. The observation here is that Simpsons system allows "exporting" the assumptions φ^y and xRy out of the subproof that occurs between the $\Box_{\mathsf{I}}$ and $\Box_{\mathsf{E}}$, to be used again later in a different "modal setting". This behaviour is what corresponds to the k_5 axiom and the context restriction prevents this. Though we formally are still allowing the assumption to be exported, it nevertheless cannot be used later in the derivation of $(\Box\varphi)^x$ as to apply $(\Box_{\mathsf{E}})$, we must use only the assumptions $\Gamma|_x$. This behaviour turns derivations between $\Box_{\mathsf{I}}$ and $\Box_{\mathsf{E}}$ into "strict subproofs" reminiscent of Jaśkowski/Fitch style natural deduction rules used for Classical Modal Logic, such as those found in [3]. Thus it is believed that this approach could also be used to bridge the gap between Jaśkowski/Fitch style presentations and Gentzen/Prawitz style presentations of natural deduction.

References

1. Ryuta Arisaka, Anupam Das, and Lutz Strassburger. On nested sequents for constructive modal logics. *Logical Methods in Computer Science*, Volume 11, Issue 3, Sep 2015.
2. Gianluigi Bellin, Valeria De Paiva, and Eike Ritter. Extended curry-howard correspondence for a basic constructive modal logic. 05 2003.
3. V.A.J. Borghuis. *Coming to terms with modal logic : on the interpretation of modalities in typed lambda-calculus*. Phd thesis 1 (research tu/e / graduation tu/e), Mathematics and Computer Science, 1994.
4. Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, Jul 2009.
5. Jim de Groot, Ian Shillito, and Ranald Clouston. Semantical analysis of intuitionistic modal logics between ck and ik, 2025.

6. Rajeev Goré and Revantha Ramanayake. Labelled tree sequents, tree hypersequents and nested (deep) sequents. *Advances in Modal Logic*, 9, 01 2014.
7. Simone Martini, Andrea Masini, and Margherita Zorzi. From 2-sequents and linear nested sequents to natural deduction for normal modal logics. *ACM Trans. Comput. Logic*, 22(3), June 2021.
8. Andrea Masini. 2-sequent calculus: Intuitionism and natural deduction. *Journal of Logic and Computation*, 3(5):533–562, 10 1993.
9. Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.