Coderivative intuitionistic logics of topological and bimetric spaces

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In a recent paper [dGS25], de Groot and Shillito proved that iS4 is sound and complete with respect to all models $\langle W, \preceq, \tau, V\rangle$, where: $\langle W, \preceq, V\rangle$ is a model for IPC, $\langle W, \tau\rangle$ is a topology, and all open sets of τ are upwards closed with respect to $\preceq$. Furthermore, they showed that completeness still holds even if we require $\preceq$ to be the specialisation order of τ. In this sense, iS4 is complete with respect to all topological models.

We extend their result in two directions. First, we prove analogous topological completeness results for various logics related to iwK4, an intuitionistic version of the logic wK4 of weak transitive frames [È01]. In our setting, we interpret the $\Box$ modality via the coderivative operator—the dual of the Cantor derivative—instead of the interior operator. We can also omit the intuitionistic relation $\preceq$—as long as we interpret it as a coderivative specialisation order. Second, we prove completeness with respect to various classes of bimetric spaces. We do not have completeness if we restrict ourselves to models consisting of an intuitionistic relation and a metrizable topological space: in this setting the ordering $\preceq$ needs to be the equality relation and so the logic becomes classical.

We now briefly sketch our results. Due to limited space, we restrict ourselves to the system iK4—the least logic containing all intuitionistic tautologies and the axiom schemas $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and $\Box\varphi \to \Box\Box\varphi$, closed under *modus ponens* and necessitation. Similar constructions work for the logics iwK4, iGL, iS4, iGrz, iCK4, IEL, PLL, mHC, iSL, and KM. As far as we are aware, iwK4 and iGrz have not been studied in the literature before. See [vdG22, Lit14] for information on the other logics.

Recall that Kripke models for IPC are tuples $\langle W, \preceq, V\rangle$ where $\preceq$ is a reflexive and transitive relation over W and $V(p)$ is upwards closed under $\preceq$, for all propositional symbols p.

Definition 1 (birelational model)**.** A birelational iK4-*model* M is a tuple $\langle W, \preceq, R, V\rangle$ where $\langle W, \preceq, V\rangle$ is a Kripke model for IPC and R is a transitive relation over W. We also require that the model satisfies the following confluence property: for all $w, v, u \in W$, if $w \preceq vRu$, then wRu. Over birelational iK4-models, the valuation of $\Box$-formulas is given by:

$$M, w \models \Box\varphi \text{ iff, for all } v \in W, \text{ if } wRv, \text{ then } M, v \models \varphi.$$

Given a topological space $\langle W, \tau\rangle$, we define the coderivative operator e_τ as follows: $w \in e_\tau(X)$ iff there is $U \in \tau$ such that $w \in U$ and $U \setminus \{w\} \subseteq X$. A subset X of W is a coderived subset iff there is $Y \subseteq W$ such that $X = e_\tau(Y)$.

Definition 2 (intuitionistic topological model)**.** An intuitionistic topological iK4-model is a tuple $M = \langle W, \preceq, \tau, V\rangle$, where $\langle W, \preceq, V\rangle$ is a Kripke model for IPC and τ is a T_d-topology over W: for all $w \in W$, $\{w\}$ is the intersection of an open and a closed set. We further require that every coderived subset of W is upwards closed under $\preceq$. We say a topological iK4-model is specialised when $w \preceq v$ iff, for all coderived $X \subseteq W$, if $w \in X$ then $v \in X$. Over topological models, the valuation of $\Box$-formulas is given by:

$$M, w \models \Box\varphi \text{ iff } w \in e_\tau(\|\varphi\|^M).$$

Definition 3 (bimetric model)**.** A bimetric iK4-model is a tuple $M = \langle W, \delta_\to, \delta_\Box, V\rangle$ such that: $\langle W, \delta_\to\rangle$ and $\langle W, \delta_\Box\rangle$ are metric spaces; $V(p)$ is $\delta_\to$-open for all p. Over bimetric models, the valuation of $\Box$-formulas is defined as

$$M, w \models \Box\varphi \text{ iff } w \in i_\to(e_\Box(\|\varphi\|)),$$

where $i_{\rightarrow}$ denotes the interior operator induced by $\delta_{\rightarrow}$ and $e_{\Box}$ denotes the coderivative operator induced by $\delta_{\Box}$.

Theorem (completeness)**.** $\mathsf{iK4}$ *is sound and complete with respect to birelational, topological, specialised topological, and bimetric models.*

Denote the class of birelational, topological, specialised topological and bimetric $\mathsf{iK4}$-models by $\mathbb{K}_{\mathsf{iK4}}$, $\mathbb{T}_{\mathsf{iK4}}$, $\mathbb{T}^s_{\mathsf{iK4}}$, and $\mathbb{M}_{\mathsf{iK4}}$, respectively. We use $\vdash$ and $\models$ to denote provability and validity, defined as usual. The proof of the above theorem proceeds as follows:

$$\begin{tikzcd}
\mathbb{T}^s_{\mathsf{iK4}} \models \varphi \arrow[d] & \mathsf{iK4} \vdash \varphi \arrow[l] \arrow[dr] & \\
\mathbb{T}_{\mathsf{iK4}} \models \varphi \arrow[r] & \mathbb{K}_{\mathsf{iK4}} \models \varphi \arrow[u] & \mathbb{M}_{\mathsf{iK4}} \models \varphi \arrow[l]
\end{tikzcd}$$

Finally, we remark on some future lines of research left open by this work. The first is to prove McKinsey–Tarski-type theorems for bimetric spaces. The second is to extend these results to logics with both diamonds and boxes. Polytopological semantics for $\mathsf{IK4}$ and $\mathsf{IS4}$ were studied in [AFDP]; it appears that specialised topological models cannot be extended to these settings. On the metric side, there is still no work done on logics with both modalities.

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