▸ SOFÍA SANTIAGO-FERNÁNDEZ, DAVID FERNÁNDEZ-DUQUE, AND JOOST J. JOOSTEN, *The Complexity of the Constructive Master Modality*.
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§1. Introduction. There are many natural ways to define modal logics over an intuitionistic base (see e.g. [5, 6, 7, 8, 9]), with axioms and frame conditions varying by application. In particular, the 'constructive' variants [10, 1] are motivated by computational considerations and provide extensions of the Curry-Howard correspondence [11]. In this setting, worlds in Kripke semantics may be viewed as stages of computation, while modal propositions correspond to types describing code available or executable at those stages. The basic constructive modal logic is known as CK [1], and its only modal axioms are two variants of the K axiom, namely, $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Box(p \to q) \to (\Diamond p \to \Diamond q)$. Wijesekera's logic WK [2] strengthens CK with the axiom $\neg\Diamond\bot$, which excludes access to inconsistent future stages and can be viewed computationally as a form of modal safety condition.

With this computational motivation in mind, it is only natural to enrich constructive logics with modalities for iteration. We therefore consider the language $\mathcal{L}^*$ defined by the grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \Box^*\varphi \mid \Diamond^*\varphi.$$

Formulas in $\mathcal{L}^*$ are interpreted over birelational CK-frames $\langle W, \preccurlyeq, R\rangle$ [2, 14], where $\preccurlyeq$ models intuitionistic entailment and R interprets modalities. From the computational motivation perspective, $\preccurlyeq$ may be viewed as context extension, while R represents transition between stages of computation. We also consider the class of *infallible* models in which no world satisfies $\bot$; these correspond to WK-models and exclude inconsistent computation stages.

In order to maintain the 'upwards persistence' of intuitionistic truth, modalities are interpreted via a combination of $\preccurlyeq$ and R. The master modality $\Box^*$ is interpreted using the reflexive-transitive closure of the composition $(\preccurlyeq; R)$, while $\Diamond^*$ is satisfied whenever every $\preccurlyeq$-accessible world admits an R-path to a world satisfying the formula. Computationally speaking, $\Box^*\varphi$ holds if φ remains valid throughout all future computations and all refinements of the current context; while $\Diamond^*\varphi$ captures a persistent form of eventuality: the possibility of achieving φ survives every future refinement of the current context.

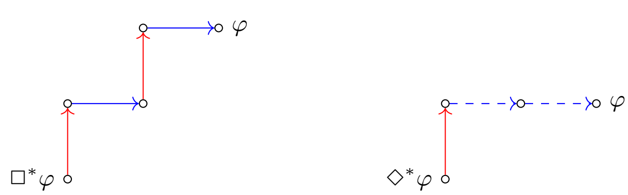

FIGURE 1. Interpretation of the master modalities. The intuitionistic relation $\preccurlyeq$ is shown in red and the modal relation R in blue. Solid arrows represent universal quantification, while dashed arrows indicate existential witnesses.

We define the constructive logics with master modality CK^* and WK^* as validity over all CK-models and WK-models, respectively. In the $\Diamond, \Diamond^*$-free fragment, we obtain the logic $\mathsf{CK}^*_\Box$, for which both Hilbert systems [12] (there denoted $\mathsf{IK}^*_\Box$) and cyclic calculi [4] (there denoted $\mathsf{IM_K}$) are known. We show that, in this fragment, validity is independent of infallibility: $\mathsf{CK}^*_\Box$ and $\mathsf{WK}^*_\Box$ coincide.

§2. Main results. We provide a chain of translations of the constructive logics with master modality into PDL, for which the finite model property and ExpTime-completeness are well known [3, 13]. Our choice of PDL as the target logic is partly motivated by the Gödel–Tarski translation provided for the intuitionistic dynamic logic iPDL into PDL, from which decidability and the finite model property are obtained [17]. Nevertheless, iPDL may best be described as only 'semi-constructive', since complex formulas do not necessarily satisfy the upwards-persistence condition of intuitionistic truth. Building on this idea, we present a translation methodology for our constructive master modality setting, summarized in Figure 2.

$$\begin{tikzcd}
\mathsf{K}^* \arrow[r, hook] & \mathsf{CK}^*_\Box \arrow[r, hook] & \mathsf{CK}^* \arrow[r, hook] & \mathsf{WK}^* \arrow[r, hook] & \mathsf{PDL} \\
& & \mathsf{CS4} \arrow[u, hook] & \mathsf{WS4} \arrow[u, hook] &
\end{tikzcd}$$

FIGURE 2. Overview of the logics with the master modality and the translations.

The language of PDL is defined by:

$$\begin{aligned}
\varphi &:= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [\alpha]\varphi \\
\alpha &:= a \mid (\alpha;\alpha) \mid \alpha^*
\end{aligned}$$

where p ranges over a set of propositional atoms and a over a set of atomic programs Π_0. In our setting we fix $\Pi_0 = \{i, m\}$, which encode respectively the intuitionistic and modal accessibility relations. We consider the classical logic with master modality K^* as the fragment of PDL over the language with a single atomic program a, whose only programs are a and a^*.

In our translation pipeline, we first show that CK^* can be faithfully reduced to WK^*. This translation allows us to work under infallibility without loss of generality and yields a simpler translation into classical logic.

We then provide a Gödel-Tarski translation from WK^* into PDL, ensuring upwards persistence of truth. The translation follows the pattern of that in [17], but the modalities require some additional care to ensure upwards persistence. The constructive modality $\Diamond$ becomes $[i^*]\langle m\rangle$, which corresponds precisely to the definition of its constructive semantics. The treatment of $\Diamond^*$ is similar, but using $[i^*]\langle m^*\rangle$ instead. We could similarly replace $\Box$ by $[i^*][m]$, but the PDL syntax allows us to instead write $[i^*;m]$. While the distinction is inessential, this is no longer the case for $\Box^*$, which *must* be interpreted as $[(i^*;m)^*]$ (or an equivalent program).

This translation enables the transfer of key properties from PDL to the constructive setting. In particular, WK^* inherits the exponential-size model property and the ExpTime upper bound for the validity problem. As a consequence, the same bounds hold for $\mathsf{CK}^*_\Box$, thereby proving conjecture of [4].

Theorem 1. *Validity of* WK^*, CK^* *and* $\mathsf{CK}^*_\Box$ *is in* ExpTime.

Theorem 2. WK^*, CK^*, *and* $\mathsf{CK}^*_\Box$ *have the exponential-size model property.*

We also obtain matching lower bounds via an embedding of the classical logic with master modality K^* into the $\Diamond$-free logic $\mathsf{CK}^*_\Box$. This is achieved through a 'converse Gödel-Tarski'-style translation that internalizes classical reasoning: the translation of a K^*-formula φ holds in a constructive model precisely when, assuming the excluded middle for all subformulas of φ at every accessible world, then φ itself holds.

Theorem 3. *Validity of* WK^**,* CK^* *and* $\mathsf{CK}^*_\Box$ *is* EXPTIME*-hard.*

As an application, we show that the constructive logic $\mathsf{CS4}$ [15] (obtained by extending CK with $\Box p \to \Box\Box p$ and $\Diamond\Diamond p \to \Diamond p$) can be translated into the $\Diamond^*, \Box^*$-fragment of CK^*. Semantically, the $\mathsf{CS4}$ modalities can be viewed as standard modal operators interpreted over the composition $(\preccurlyeq; R)$, which naturally suggests a translation interpreting $\Box$ and $\Diamond$ by the master modalities $\Box^*$ and $\Diamond^*$, respectively. Hence, $\mathsf{CS4}$ is also decidable in EXPTIME, improving the previously known NEXPTIME upper bound obtained via finite model arguments [16]. The same method yields an EXPTIME upper bound for the Wijesekera variant $\mathsf{WS4}$.

Theorem 4. *Validity for* $\mathsf{CS4}$ *and* $\mathsf{WS4}$ *is in* EXPTIME.

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